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The Extremal Graph Problem of the Icosahedron

MIKLÓS SIMONOVITS

University L. Eötvös, Budapest, Hungary

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P. Turán has asked the following question:

Let I^{12} be the graph determined by the vertices and edges of an icosahedron. What is the maximum number of edges of a graph G^n of n vertices if G^n does not contain I^{12} as a subgraph?

We shall answer this question by proving that if n is sufficiently large, then there exists only one graph having maximum number of edges among the graphs of n vertices and not containing I^{12}. This graph H^n can be defined in the following way:

Let us divide $n - 2$ vertices into 3 classes each of which contains $[(n - 2)/3]$ or $[(n - 2)/3] + 1$ vertices. Join two vertices iff they are in different classes. Join two vertices outside of these classes to each other and to every vertex of these three classes.

NOTATIONS

The graphs considered in this paper have neither loops nor multiple edges, they will be denoted by capital letters, and the upper indices will always denote the number of their vertices. The vertices will be denoted by $x, y,...,$ the edges by $(x, y),...$. The number of vertices, edges, and the chromatic number of the graph G will be denoted by $v(G)$, $e(G)$, and $\chi(G)$, respectively. If $x \in G$, stx denotes the star of x, i.e., the set of vertices, joined to x. The cardinality of stx, i.e., the valence of x, will be denoted by $\sigma(x)$. The cardinality of the set E will be denoted by $|E|$.

To simplify the definitions of some special graphs, we introduce the following operations:

Sum, product. Let us suppose that $G_1,..., G_d$ are graphs without common vertices. Their disjoint union will be called their sum and denoted by $\sum G_p$; joining each vertex of G_i to each one of G_j for every $1 \leqslant i < j \leqslant d$, we get their product denoted by $\bigtimes G_p$. If $G_1 \times G_2$ is a subgraph of G, i.e., each vertex of $G_1 \subset G$ is joined to each one of $G_2 \subset G$, then we shall say that G_1 is completely joined to G_2 (in G).

Difference. If G_1 is a subgraph of G or simply a set of vertices and edges of G, then $G - G_1$ denotes the graph obtained from G by omitting all the edges and vertices of G_1 and all the edges at least one endpoint of which belongs to G_1.

Special graphs. $K_d(r_1, ..., r_d)$ is the complete d-partite graph with r_p vertices in its pth class. In particular, $K_d(1, ..., 1) = K_d$ is the complete graph of d vertices and $K_1(d)$ is its complementer: d independent vertices. P^k and C^k denote the path and circuit of k vertices, respectively.

1. Introduction

A well-known theorem of P. Turán [1] (1941) asserts that for given p and $n \geqslant p$, there exists exactly one graph S_o^n having maximum number of edges among the graphs G^n not containing K_p. Further:

$$S_o^n = K_{p-1}(n_1, ..., n_p), \tag{1}$$

where

$$\sum n_i = n \qquad \text{and} \qquad \left| n_i - \frac{n}{p-1} \right| \leqslant 1. \tag{1*}$$

(The integers n_i are uniquely determined by (1*) apart from a permutation, therefore (1) and (1*) determine S_o^n uniquely.)

One can replace K_p in Turán's problem by any given graph and ask: For given L, what is the maximum number of edges a graph G^n can have if it does not contain L as a subgraph? P. Erdös and the author have fairly general results in this case, the most important of which is

THEOREM A. *If $\chi(L) = p$, then every extremal graph S^n for L, i.e., every graph having maximum number of edges among graphs of n vertices not containing L, can be obtained from S_o^n (defined by* (1) *and* (1*)) *by omitting $o(n^2)$ edges and adding $o(n^2)$ new edges.*

Theorem A gives quite a lot of information about the extremal graphs; however, if we wish to find the exact structure of the extremal graphs, we need more information than the chromatic number of L. The problems we meet are often hopeless.

P. Turàn asked (Erdös published [8]) the following question: What is the maximum number of edges and what are the extremal graphs in the case if L is a graph determined by the vertices and edges of a regular polyhedron. The tetrahedron-graph is just K_4, therefore the answer in this case is given by Turán's theorem. In the case of the octahedron and cube, P. Erdös and the author have some results [3, 4]. Let

$$H(n, d, s) = K_{s-1} \times K_d(n_1, \ldots, n_d), \tag{2}$$

where

$$\sum n_p = n - s + 1 \quad \text{and} \quad \left| n_p - \frac{n-s+1}{d} \right| < 1. \tag{2*}$$

According to [5], if $n > n_0$, $H(n, 2, 6)$ is the only extremal graph for the dodecahedron-graph. Let Γ^{12} be the icosahedron-graph.

The main purpose of this paper is to prove the following result:

THEOREM 1. *There exists an n_0 such that $n > n_0$, then $H(n, 3, 3)$ is the only extremal graph for Γ^{12}.*

Remark. One can conjecture that for $n > 17$, $H(n, 3, 3)$ is always an extremal graph for Γ^{12} and that there are no other extremal graphs for Γ^{12} if $n > 20$. However, for $n = 16$, $H(n, 3, 3)$ is not an extremal graph, since $K_5 \times K_2(6, 5)$ does not contain Γ^{12} and has more edges than $H(16, 3, 3)$. For $n = 19$, $H(19, 3, 3)$ cannot be the only extremal graph (if it is an extremal graph at all) because

$$(K_5 + K_2) \times K_2(6, 6)$$

has the same number of edges and does not contain Γ^{12} either. (Theorem 1 was first stated in the Ph.D. thesis of the author [9].)

2. GRAPHS CONTAINING Γ^{12}

Since the structure of Γ^{12} is rather complicated, it would be difficult to verify directly that a graph G^n contains Γ^{12}. Therefore we shall define three graphs W_1, W_2, and W_3 having simpler structure and containing Γ^{12}. In our arguments, wishing to prove something about the extremal graph S^n (for Γ^{12}), we shall suppose the contrary, prove that in this case S^n contains one of W_1, W_2, W_3 and therefore Γ^{12} as well, and this will be a contradiction.

Let us have a look at Fig. 1. Both graphs are the same, Γ^{12}. The reader can easily check that Γ^{12} does not contain 4 independent vertices. Therefore, $\chi(\Gamma^{12}) \geqslant 12/3 = 4$. On Fig. 2 we give a coloring of Γ^{12} with 4 colors. This shows that $\chi(\Gamma^{12}) = 4$.

At the same time the coloring given on Fig. 2 proves that

$$\Gamma^{12} \subset P^6 \times K_2(3, 3) = W_1. \tag{3}$$

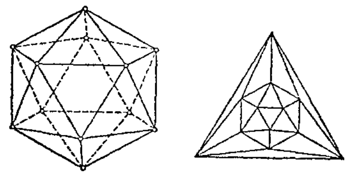

FIGURE 1

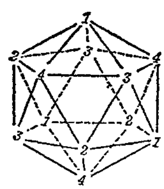

FIGURE 2

Indeed, on Fig. 3 a solid line indicates the edges joining vertices of color 1 to vertices of color 2 (on Fig. 2). We can see that these edges determine a P^6. Since each color is used 3 times, (3) is proved.

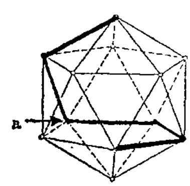

FIGURE 3

We prove that

$$\Gamma^{12} \subset (P^3 + P^2) \times (K_2 + K_1(2)) \times K_1(3) = W_2 . \tag{4}$$

Indeed, if we change the color of the vertex a (see Fig. 3) from the original 1 to 4, then the graph spanned by the vertices of color 1 and 2 will be just the union of a P^3 and an edge, the graph spanned by the vertices of color 4 will become a graph of 4 vertices and 1 edge, and finally, the vertices of color 3 will remain independent. This proves (4).

Omitting the vertices of two opposite triangles of Γ^{12} (see Fig. 4), we obtain just a circuit of length 6. Therefore

$$\Gamma^{12} \subset (K_3 + K_3) \times K_2(3, 3) = W_3 . \tag{5}$$

3. Proof of Theorem 1

There are several ways to prove Theorem 1 but none of those which I know is simple. After long hesitation, I decided to publish the shortest

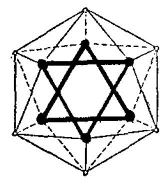

FIGURE 4

one which is based on Theorems 1 and 2 of [5]. However, the proof of these theorems (of [5]) are rather long and involved; therefore the proof given here cannot be called elementary.

First, I will define a class of "very symmetrical" graphs, then formulate Theorem B, which is a weakened form of Theorems 1 and 2 of [5].

DEFINITION 1. *Symmetric subgraphs.* Let T_1 and T_2 be connected spanned subgraphs of G. They are called symmetric if either $T_1 = T_2$ or

(i) $T_1 \cap T_2 = \varnothing$, and

(ii) $(x, y) \notin G$ if $x \in T_1$, $y \in T_2$, and

(iii) there exists an isomorphism ω_2: $T_1 \to T_2$ such that for every $x \in T_1$, $u \in G - T_1 - T_2$, x is joined to u if and only if $\omega_2(x)$ is joined to u.

$T_1, \ldots, T_k$ are symmetric if for every $1 \leqslant i < j \leqslant k$, T_i and T_j are symmetric.

Remarks. The transitivity of our relation is the consequence of the connectedness of the considered graphs.

Speaking about a family of symmetric graphs we always suppose that the isomorphisms ω_i: $T_1 \to T_i$ are fixed, even if they are not uniquely determined. The vertices x and $\omega_i(x)$ will be called corresponding vertices.

DEFINITION 2. $\mathscr{Q}(n, r, d)$ is the class of graphs G^n having the following properties:

(i) It is possible to omit $\leqslant r$ vertices of G^n so that the remaining graph G^* is a product:

$$G^* = \bigtimes_{p \leqslant d} G^{m_p} \qquad \text{where} \quad \left| m_p - \frac{n}{d} \right| \leqslant r.$$

(ii) For every $p \leqslant d$, there exist connected graphs $H_{p,j} \subset G^{m_p}$ and isomorphisms $\omega_{p,j}$: $H_{p,1} \to H_{p,j}$ such that $v(H_{p,j}) \leqslant r$ and $H_{p,j}$ $(j = 1, \ldots, \rho_p)$ are symmetric subgraphs of G^n and $G^{m_p} = \sum_j H_{p,j}$.

THEOREM B. *Let* $\chi(L) = d + 1$ *and* $v(L) = \tau$. *If*

$$L \subset P^{\tau} \times K_{d-1}(\tau, \ldots, \tau), \tag{6}$$

then there exists a constant $r = r(L)$ *such that for every* n, $\mathcal{Q}(n, r, d)$ *contains an extremal graph for* L. *Furthermore, if there exists an* n_0 *such that for every* $n > n_0$ $\mathcal{Q}(n, r, d)$ *contains only one extremal graph, then for sufficiently large values of* n *this is the only extremal graph.*

Proof of Theorem 1. We apply Theorem B to the icosahedron with $d = 3$. This is allowed according to (3). Now the only thing we have to prove is that for every r if $S^n \in \mathcal{Q}(n, r, 3)$ is an extremal graph for Γ^{12} and $n > n_0(r)$, then $S^n = H(n, 3, 3)$. Since $S^n \in \mathcal{Q}(n, r, 3)$, we can omit a set U of vertices of S^n such that the remaining

$$S^n - U = \bigtimes_p \sum_j H_{p,j},$$

where $v(H_{p,j}) \leqslant r$ and $|U| < r$ and for every $p \leqslant 3$ the family $H_{p,j}$ is symmetric in S^n. We may suppose that

$$v(H_{1,1}) \geqslant v(H_{2,1}) \geqslant v(H_{3,1}). \tag{7}$$

Let B_p be the set of vertices of $\sum_j H_{p,j}$ $(p = 1, 2, 3)$. Clearly, if $m = n/3$, then

$$|B_p| = m_p = n/3 + O(1) = m + O(1). \tag{8}$$

(A) First we give an upper bound of $e(S^n)$. Let $x_1, \ldots, x_q$ be the vertices of U having valence $\geqslant 2m$. Let $\delta_i = \sigma_i - 2m$. Then

$$0 \leqslant e(S^n) - e(H(n, 3, 3)) \leqslant \sum \delta_i - 2m + \sum_p \sum_j e(H_{p,j}) + O(1). \tag{9}$$

To prove (9) let us omit all the edges of $\sum \sum H_{p,j}$ from S^n and also all the edges starting from U. Then, let us join each vertex of U to each one of $B_2 \cup B_3$. Finally, select two vertices from B_1 and join them to all the other vertices of B_1. Thus we increased the number of edges of S^n by at least

$$2m - \sum \delta_i - \sum\sum e(H_{p,j}) + O(1). \tag{10}$$

The obtained graph G^n has the following property:

It is possible to omit two vertices of G^n so that the resulting graph be 3-chromatic. (11)

It is easy to check that among all the graphs of n vertices satisfying (11), there exists exactly one having maximum number of edges, namely

$H(n, 3, 3)$. Therefore,

$$e(G^n) \leqslant e(H(n, 3, 3)).$$

This and (10) prove the right-hand side of (9). To prove the left-hand side of (9), it is enough to prove that

$$H(n, 3, 3) \text{ does not contain } I^{12}. \tag{12}$$

Omitting any two vertices of I^{12}, we obtain a graph of 10 vertices which does not contain 4 independent vertices. Therefore, this obtained graph is (at least) 4-chromatic, i.e., I^{12} does not satisfy (11). On the other hand, $H(n, 3, 3)$ satisfies (11) and thus every subgraph of it must also satisfy (11). Hence, $I^{12} \not\subset H(n, 3, 3)$, which proves that

$$e(S^n) \geqslant e(H(n, 3, 3)).$$

(B) Our next purpose is to prove that

$$v(H_{2,1}) = v(H_{3,1}) = 1. \tag{13}$$

Let us suppose the contrary. Then (because of (7)),

$$v(H_{1,1}) \geqslant v(H_{2,1}) \geqslant 2. \tag{14}$$

Trivially,

$$\begin{aligned} S^n &\supset (H_{1,1} + H_{1,2}) \times (H_{2,1} + H_{2,2}) \times K_1(3) \\ &\supset (H_{1,1} + H_{1,2}) \times (K_2 + K_1(2)) \times K_1(3). \end{aligned}$$

Since S^n does not contain I^{12}, it does not contain W_2 either (see (4)) and therefore $H_{1,1} + H_{1,2}$ does not contain $P^3 + P^2$, i.e., $H_{1,1}$ does not contain P^3. $H_{1,1}$ is connected, thus $v(H_{1,1}) \leqslant 2$. According to (7) $v(H_{2,1})$ and $v(H_{3,1})$ are also at most 2. Hence,

$$\sum\sum e(H_{p,j}) \leqslant 3m/2 + O(1). \tag{15}$$

Hence at least one δ_i is $\geqslant m/2r$. But this can happen only if the corresponding x_i is joined to, say, all the vertices of B_2 and B_3 and at least one vertex of $H_{1,j}$ (since the graphs $H_{p,j}$ are symmetric and therefore any vertex is joined either to all the vertices of B_p or to the half of them or to none of them). But in this case x_1, $H_{1,1}$, and $H_{1,2}$ would determine a $P^3 + P^2$ completely joined to $H_{2,j}$. This $P^3 + P^2$, $H_{2,1}$, $H_{2,2}$, and 3 arbitrary vertices from B_3 would determine a W_3 in S^n. This contradiction proves (13).

(C) We prove that

(a) $v(H_{1,1}) = 1$,

(b) U contains two vertices completely joined to $S^n - U$; the other vertices are joined to all the vertices of two of B_1, B_2, B_3, and to none of the third one.

We shall need the following lemma, which is a particular case of a theorem of P. Erdös and T. Gallai (Theorem 2.6 of [7]).

LEMMA. *If G^q does not contain P^6, then*

$$e(G^q) \leqslant 2q,$$

and the equality holds iff G^q is the union of disjoint K_5's. In particular, the strict inequality holds if $K_3 \not\subset G^q$.

(This case of the Erdös–Gallai Theorem can be proved much easier than the general one.) S^n does not contain Γ^{12}, therefore it does not contain W_1 or W_3 either, and hence $H_{1,1}$ cannot contain a K_3 or a P^6. Thus, according to the Lemma,

$$\sum\sum e(H_{1,j}) \leqslant \left(2 - \frac{1}{r}\right) m + O(1).$$

Hence in (9) at least one δ_i is larger than cm if $c = (2r)^{-2}$. Because of (13), the corresponding x_i has to be joined to B_2 and B_3 and to at least cm vertices of B_1. Let $a_j \in H_{1,j}$ be corresponding vertices of the symmetric graphs $H_{1,j}$ joined to x_i. Each edge of $H_{1,j}$ must contain a_j, otherwise $H_{1,j}$ would contain a path (c_j, b_j, a_j) and

$$(c_1, b_1, a_1, x_i, a_2, b_2, c_2) = P^7$$

would be a path completely joined to B_2 and B_3. Thus S^n would contain a W_1 which is a contradiction. Therefore, a_j really is contained by each edge of $H_{1,j}$.

(C_1) First we prove that there exists at least another vertex in U joined to every vertex of B_2 and B_3 and to some vertices of $H_{1,1}$. Let us suppose the contrary. Since each edge of $H_{1,1}$ contains a_1, $H_{1,1}$ is a tree. Therefore

$$\sum\sum e(H_{p,j}) + \sum \delta_i \leqslant m\left(1 - \frac{1}{r}\right) + m + O(1). \tag{16}$$

This contradicts (9). Therefore there exists another x_k joined to some vertices of $H_{1,1}$ and to all the vertices of B_3 and B_3.

(C_2) Let a_1 and b_1 be distinct vertices of $H_{1,1}$, and let us suppose that a_1 is joined to x_i, b_1 is joined to x_k. Then there exists a P^6 in $B_1 \cup U$ completely joined to $B_2 \cup B_3$. Indeed, let us go from a_1 to b_1 in $H_{1,1}$ and denote this path by P^*. Let P^{**} be the path determined by the corresponding vertices of $H_{1,2}$. If a_2 and b_2 correspond to a_1 and b_1, respectively, then

$$(a_1P^*b_1x_kb_2P^{**}a_2x_i)$$

is a path of at least 6 vertices completely joined to B_2 and B_3. But this means that S^n contains a W_1, a contradiction. This proves that a_1 and b_1 cannot be distinct, i.e., both x_i and x_k are joined to the same (and to only one) vertex of $H_{1,1}$. If there were a third vertex x_q joined to all the vertices of B_2 and B_3 and to some vertices of $H_{1,1}$, then it would be joined to the same vertex of $H_{1,1}$ and $B_1 \cup U$ would contain a $K_2(3, 3)$ completely joined to each vertex of B_2 and B_3. Because of $P^6 \subset K_2(3, 3)$, S^n would again contain a W_1. This contradiction proves that there exist exactly two vertices of U joined to every vertex of B_2 and B_3 and to some vertices of B_1. These vertices are joined to exactly one vertex of $H_{1,1}$.

(C_3) Again, if $v(H_{1,1}) \geqslant 2$ held, then (16) would be valid, contradicting (9). Therefore $\nu(H_{1,1}) = 1$.

(C_4) We have to show that if $x \in U - \{x_i, x_k\}$, then x is joined to exactly two of B_1, B_2 and B_3. We know that it can be joined to at most two of them. On the other hand, if x were joined, e.g., only to B_3, then its valence would be $m + O(1)$. According to the general theorems of [2b,c], the minimum valence in S^n is $2n/3 + o(n)$. This proves our assertion. (If we wish to avoid using this latest theorem, we can modify (9) by taking δ_i not only for those vertices, for which δ_i is positive. Now, if there were a vertex of valence $m + O(1)$ in U, then the middle of (9) would decrease by $m + O(1)$, leading to a contradiction.)

(D) Let now C_p be the class of vertices of $U - \{x_i, x_k\}$ not joined to B_p (but joined to B_q if $p \neq q$). Let $A_p = B_p \cup C_p$ and $|A_p| = n_p$. Let $H^n = K_2 \times K_3(n_1, n_2, n_3)$. Since H^n satisfies (11),

$$e(H(n, 3, 3)) \geqslant e(H^n), \tag{17}$$

and the equality holds iff $H^n = H(n, 3, 3)$. Let D_p be the set of vertices of A_p not joined either to x_i or x_k. We show that a vertex of A_p cannot be joined to a vertex of $A_p - D_p$. Indeed, if $a \in A_p$ were joined to a $b \in A_p - D_p$, then each vertex of the path $(a, b, x_i, b^*, x_k, b^{**})$ would be joined to each vertex of B_q if $q \neq p$, where b^* and b^{**} are two arbitrary vertices of B_p. This would assure a W_1 in S^n. Therefore the vertices of A_p

really cannot be joined to $A_p - D_p$. Let $G(D_p)$ denote the graph spanned by the vertices of D_p, and let M denote the number of pairs (u, v) such that u and v belong to different A_ps and are not joined. (These pairs will be called missing edges.) Now we prove that

$$e(S^n) \leqslant e(H^n) - \sum (2 \mid D_p \mid - e(G(D_p))) - M; \tag{18}$$

equality holds iff x_i and x_k are joined in S^n and every vertex joined to at least one of x_i and x_k is also joined to the other. Indeed, if we omit all the edges of $G(D_p)$ for $p = 1, 2, 3$, and join the vertices of D_p to x_i and x_k, further join all the M pairs of vertices, called missing edges, then we obtain from S^n, H^n or $H^n - (x_i, x_k)$.

Because of the Lemma, $2 \mid D_p \mid - e(G(D_p))$ is always nonnegative. Hence

$$e(S^n) \leqslant e(H^n) \leqslant e(H(n, 3, 3)). \tag{19}$$

Since $e(S^n) \geqslant e(H(n, 3, 3))$, in (19) we have in both cases the equality. Therefore,

(a) $H^n = H(n, 3, 3)$, and

(b) $(x_i, x_k) \in S^n$, $M = 0$, and in the lemma we also have equality. The equality in the lemma implies that either $D_p = \varnothing$ or $G(D_p)$ is the union of disjoint K_5's. The latest case is excluded, since, if $G(D_1)$ contained a K_3, then this K_3 and $K_3 = (x_i, x_k, b)$ (with a b from B_1) would form a W_3 with 3 vertices from each of B_2 and B_3. Therefore $D_p = \varnothing$. This means that $S^n = H^n$ and, consequently, $S^n = H(n, 3, 3)$. Thus the proof of Theorem 1 is complete.

In fact, we have proved the following result, which is somewhat stronger than Theorem 1.

THEOREM 2. *If n is large enough, then $H(n, 3, 3)$ has more edges than any other graph G^n not containing either of the graphs W_1, W_2, and W_3.*

With a little more care we could prove a more general theorem but its formulation is too complicated, therefore we did not deal with it.

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