On the number of high multiplicity points for 1-parameter families of curves

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Happy birthday, Tom

One of the very influential results is the **Szemerédi-Trotter** theorem on incidences:

There exists a constant $c > 0$ for which, if
Given m points and n straight lines, the number of incidences between them is less than $cn^{2/3}m^{2/3} + c(n+m)$.

One generalization, Pach-Sharir:

Let $\mathcal{P}$ be a set of n points, and $\mathcal{C}$ be a set of m simple curves having k degrees of freedom and multiplicity type s, then the number of incidences between them is

$$I(\mathcal{P},\mathcal{C}) < c(s,k)\left(n^{\frac{k}{2k-1}}m^{\frac{2k-2}{2k-1}} + n + m\right)$$

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where

- k degrees of freedom = through any k points at most s curves

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and

- multiplicity type s

  = any two curves intersect in at most s points.

Spencer-Szemerédi-Trotter, Pach-Sharir, Székely.

Many other related results…

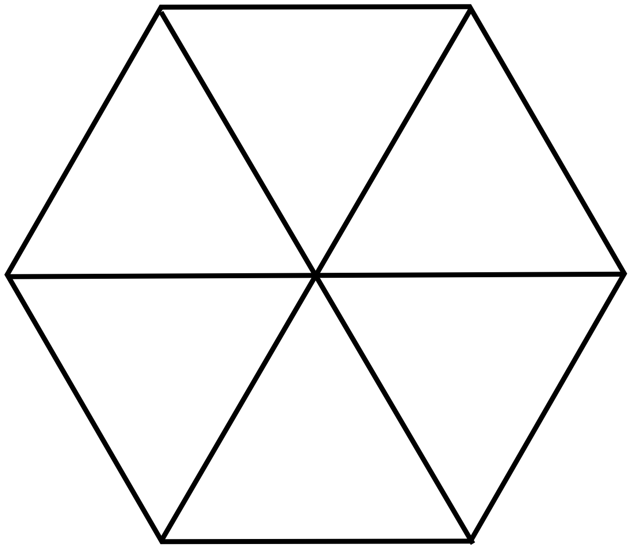
I mentioned *this* since ***this* is related the most to our topic**.

Behind everything I shall speak about is the **Elekes-Szabó theorem** and its proof heavily uses these theorems.

I shall return to the **Elekes-Szabó theorem** later.

How to distinguish

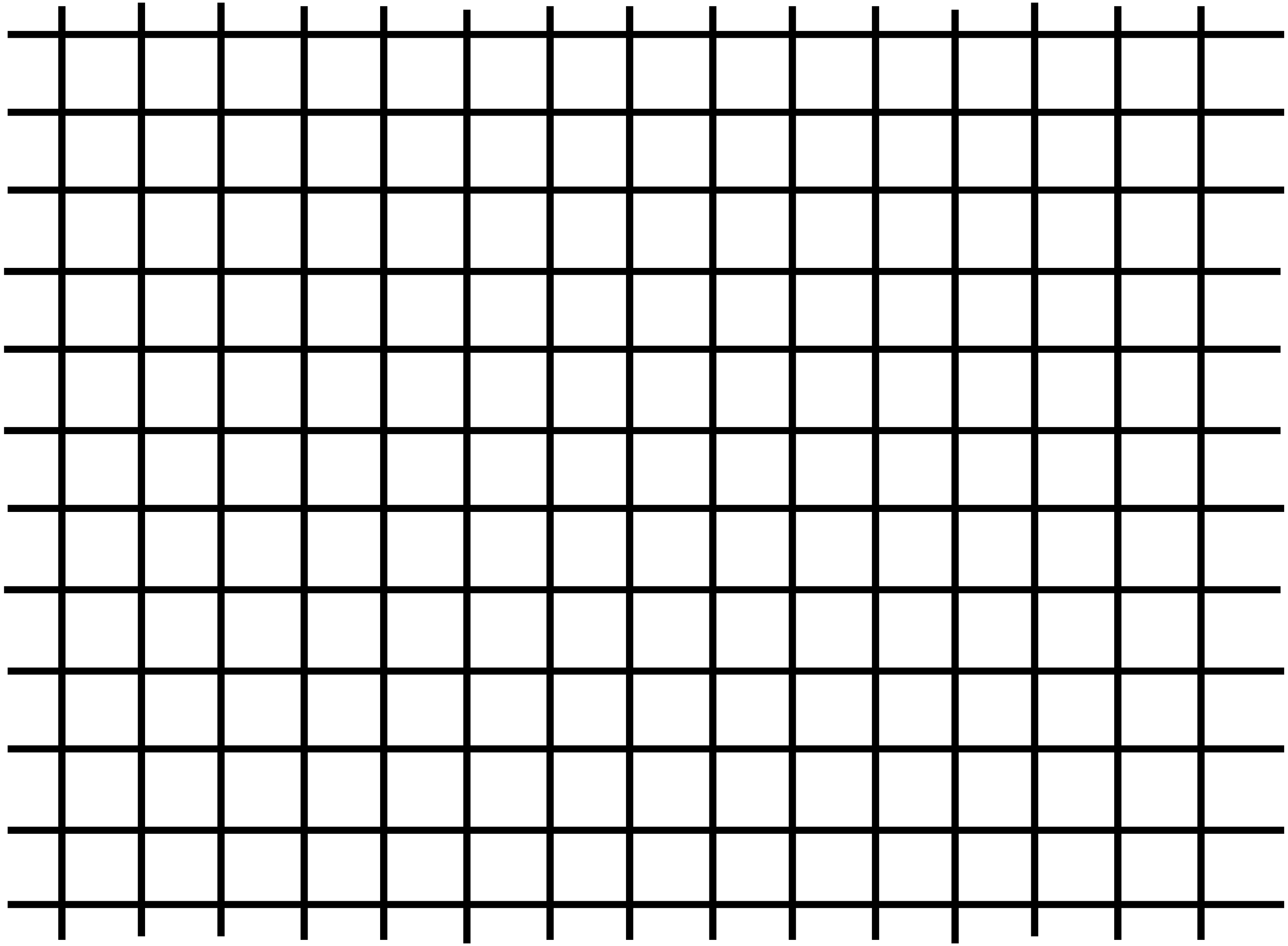
unit circles and straight lines, **combinatorially**?



Combinatorially = using the incidence matrix = using the intersection pattern.

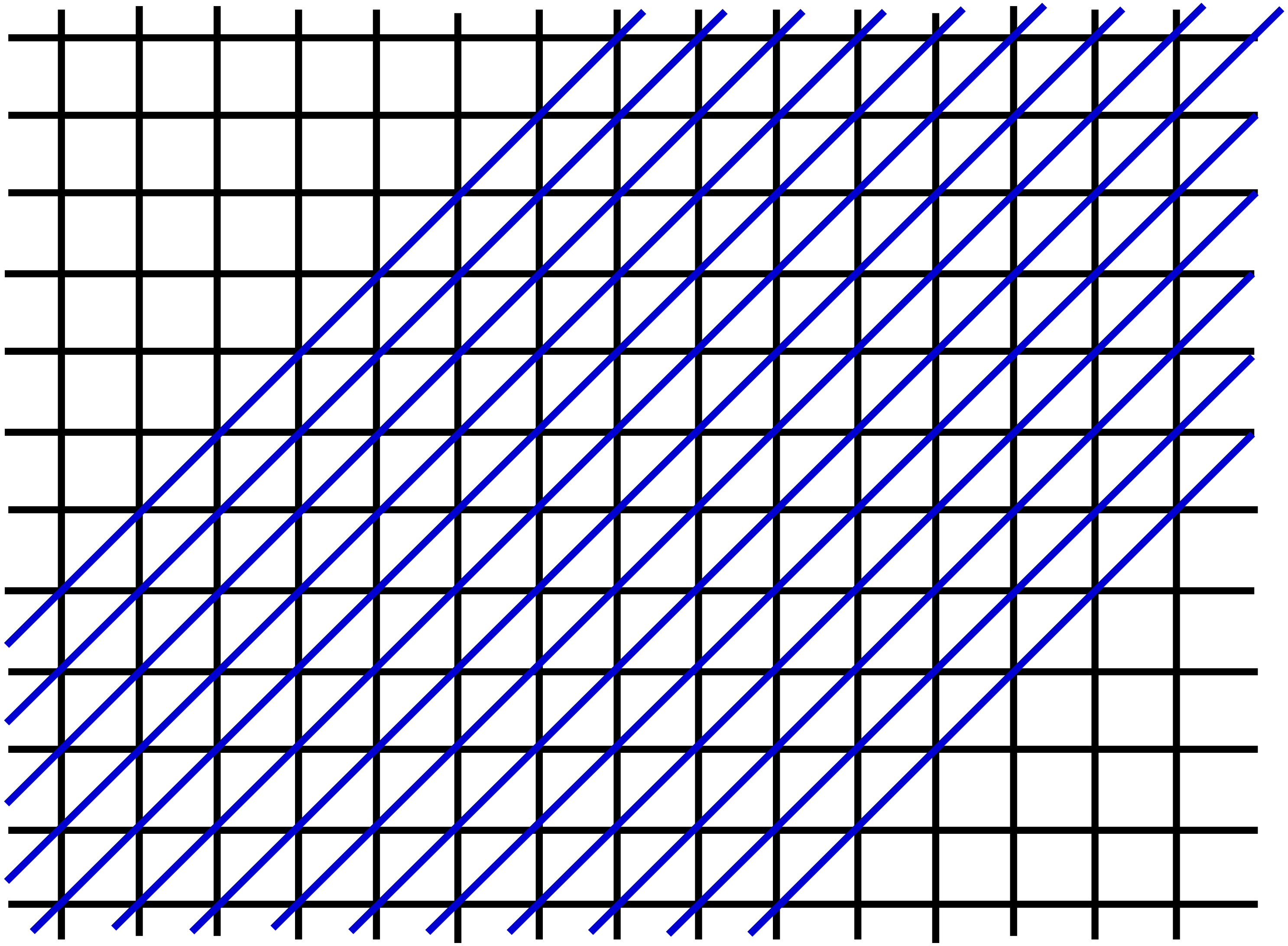
Can these curves be circles of some huge fixed radius?
(Can one have such a picture with arcs of unit circles?)

The original problem



Quadratic number of crossings

Three families



Still quadratic number of **triple crossings**

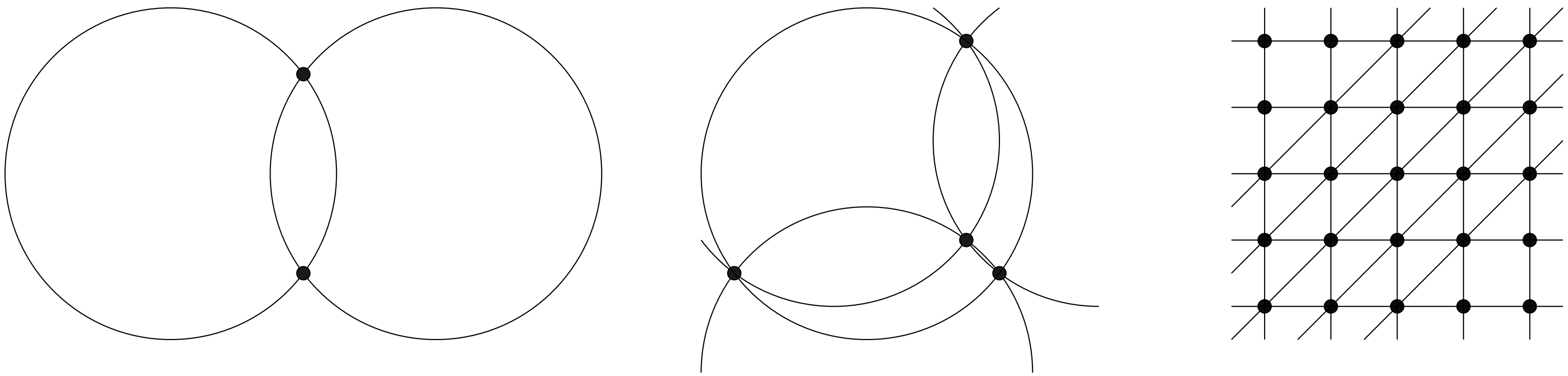
A Combinatorial Distinction between Unit Circles and Straight Lines

(a) How can one distinguish unit circles and straight lines combinatorially?

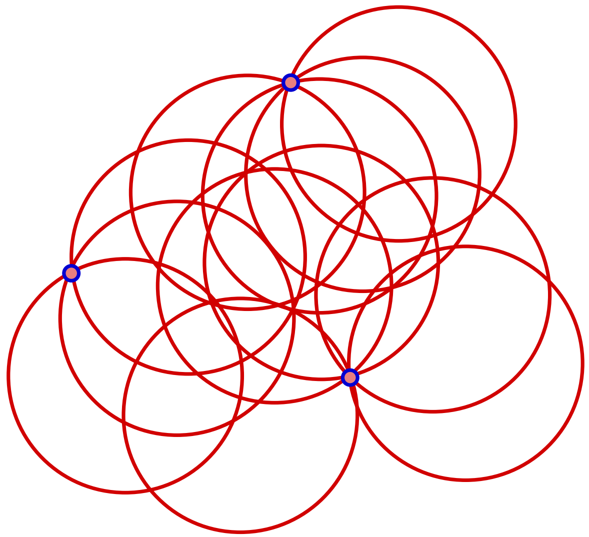
(b) Can one have as many triple points for circles as for straight lines?

$\longrightarrow$ **YES**, because of “inversion”: all patterns can be obtained by straigth lines can also be obtained by arcs of circles.

(c) Can one have as **many triple points** for unit circles as for straight lines?



Three families of circles



We consider 3 families of **unit circles**, through 3 points.

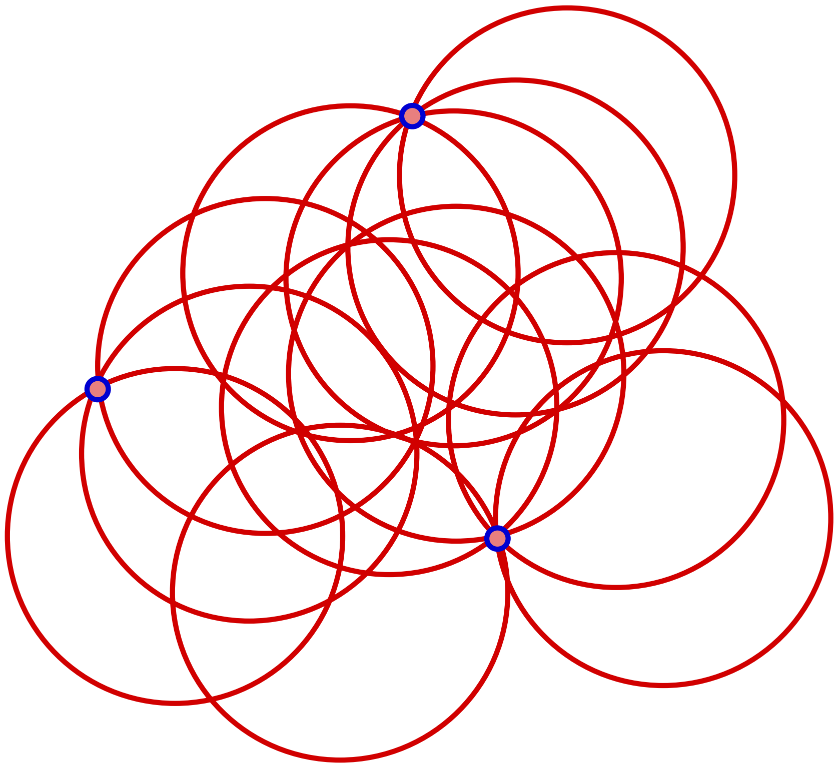
On this figure we do no see (?) triple points (except the obvious 3 points): the circles are selected at random.
Could we have cn^2 triple points?

No: we can have at most $n^{2-\eta}$ triple points.

Theorem on unit circles

There exists an $\eta > 0$ such that

If we take 3 points in the plane: A, B, C, and n unit circles through each of them, then they can determine at most $cn^{2-\eta}$ triple points.



Is there a general theorem behind this?

(a) High dimension?

Question. Is it true that:
If in some dimension d we have some surfaces of dimension s, then they cannot have two many points that are intersections of t of our surfaces?

Non-degeneracy condition is needed: consider many planes containing the same staight line: we have infinitely many triple points.

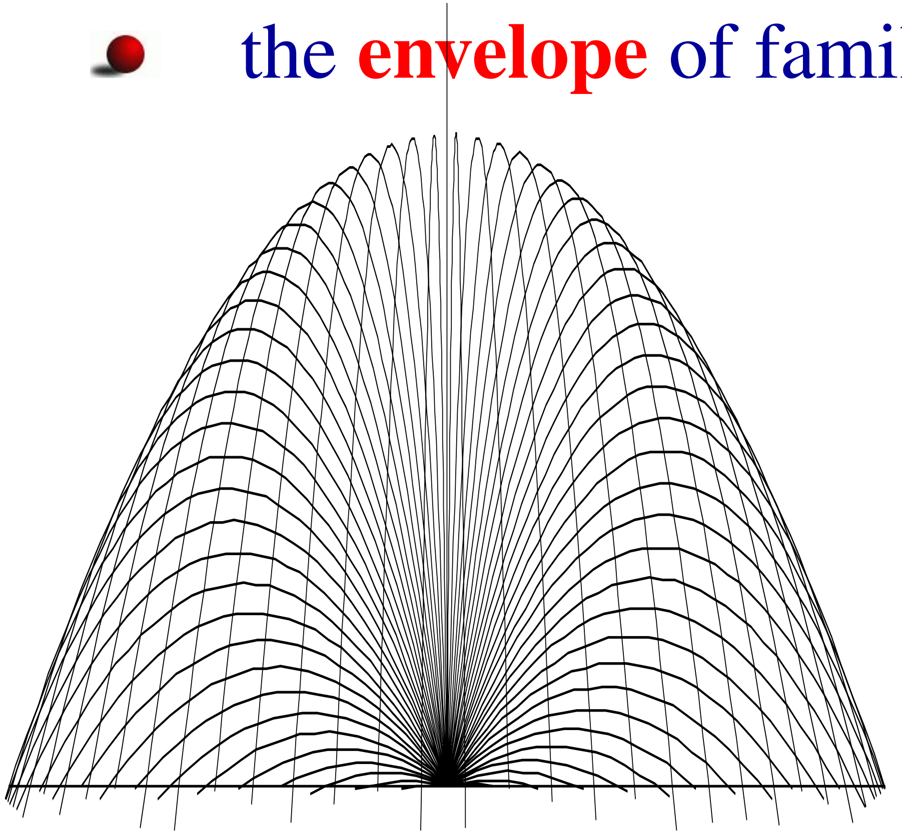
We shall restrict ourselves to curves in dimension 2.

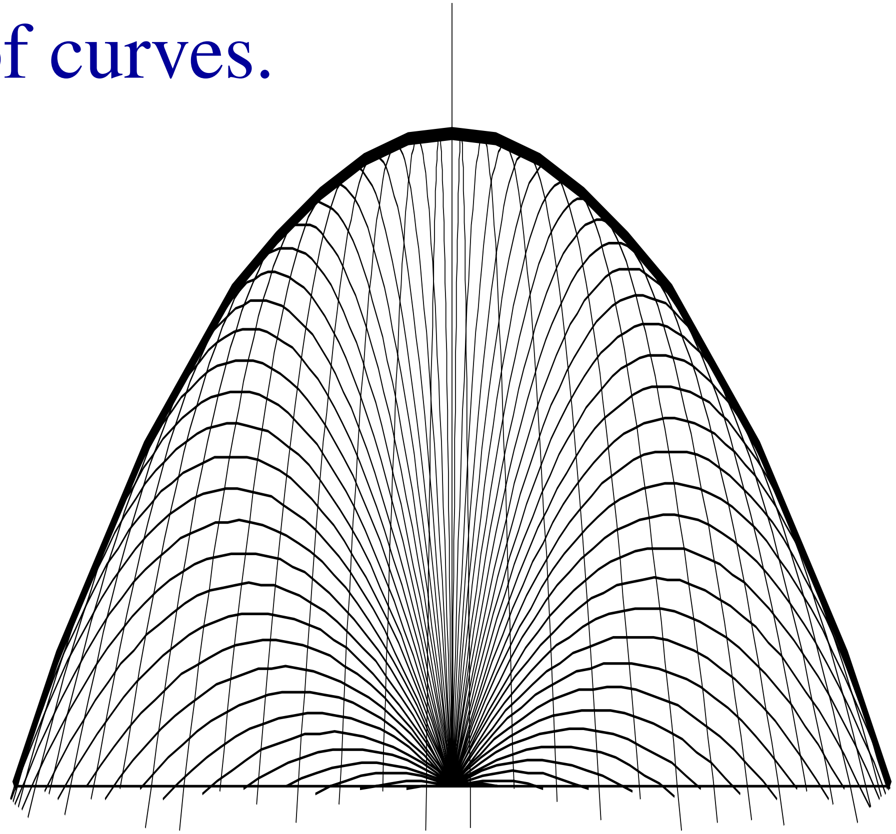
Many cases can be reduced to the case of $\mathbb{R}^2$ by projecting the high-dimensional configurations to $\mathbb{R}^2$.

The envelope

Crucial in our results:

- the **envelope** of families of curves.





- 1-parameter families
- Implicitely, analytically parametrized families

Analytic description of the envelope

The enveloping curve is in some sense a singularity:

$F(x, y, t) = 0.$	$\leftarrow$ analytic parametrization
$F_t(x, y, t) = 0.$	$\leftarrow$ partial derivative by t

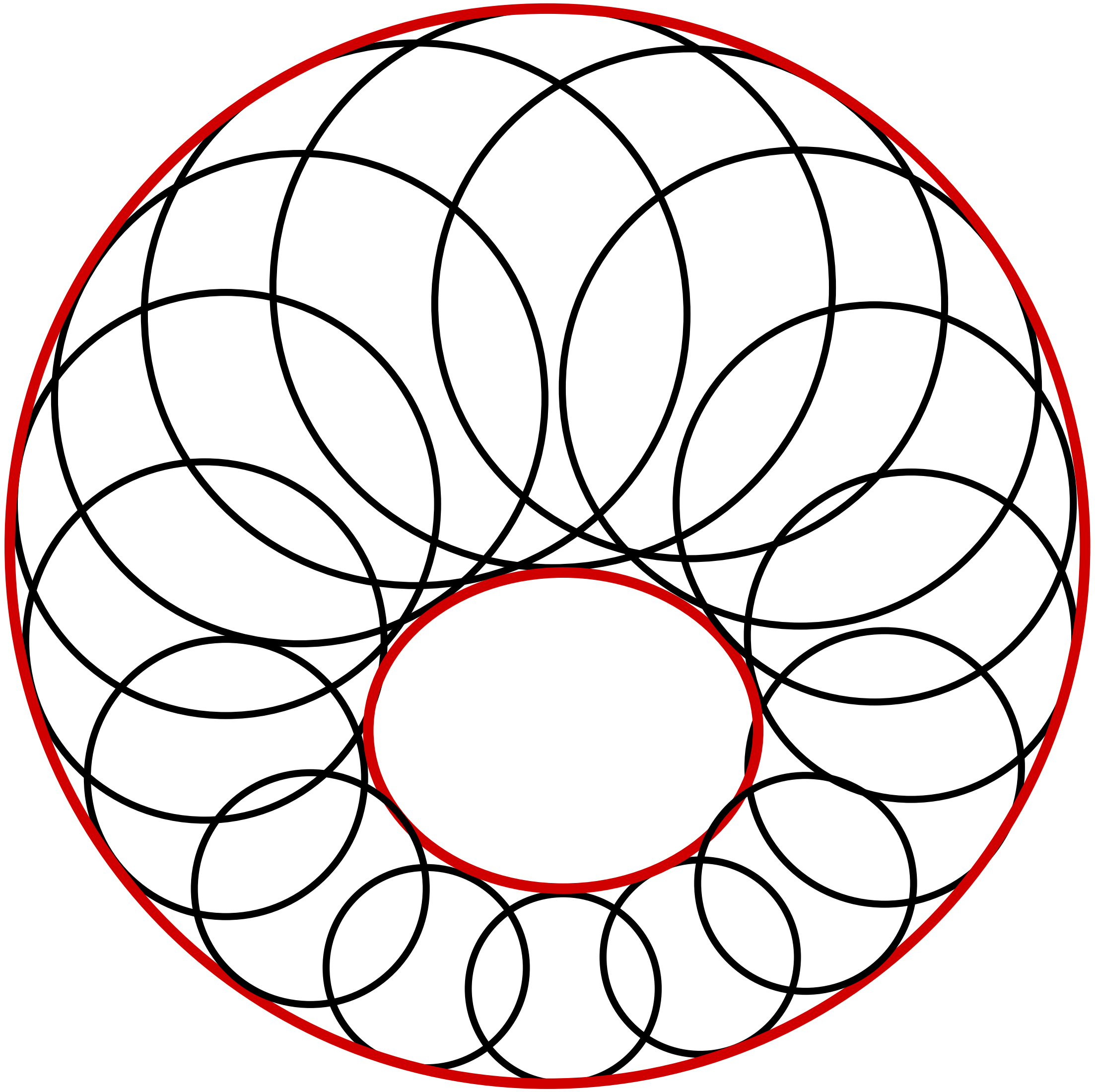
- Often we can eliminate t, getting an enveloping curve

$$\Phi(x, y) = 0.$$

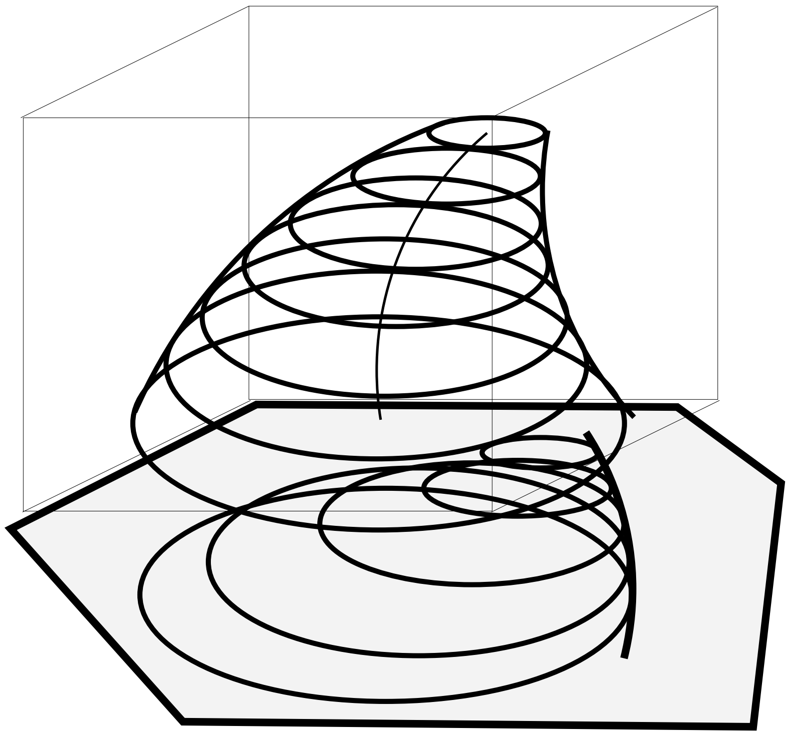
- Problems with the **analytic branches**
- If you are uncomfortable with analytic functions and branches, think of polynomials or algebraic functions defined by them.

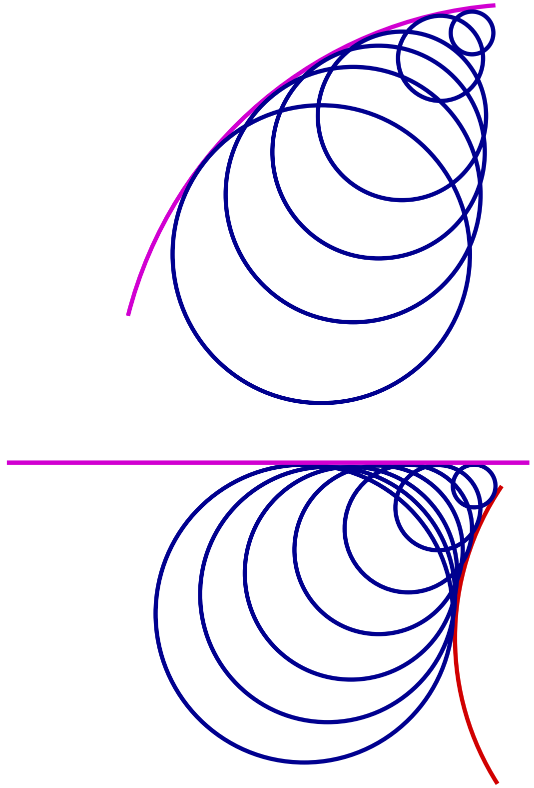
Watch out: we speak of functions, not curves, the parametrization influences our statements!

Envelope of circles



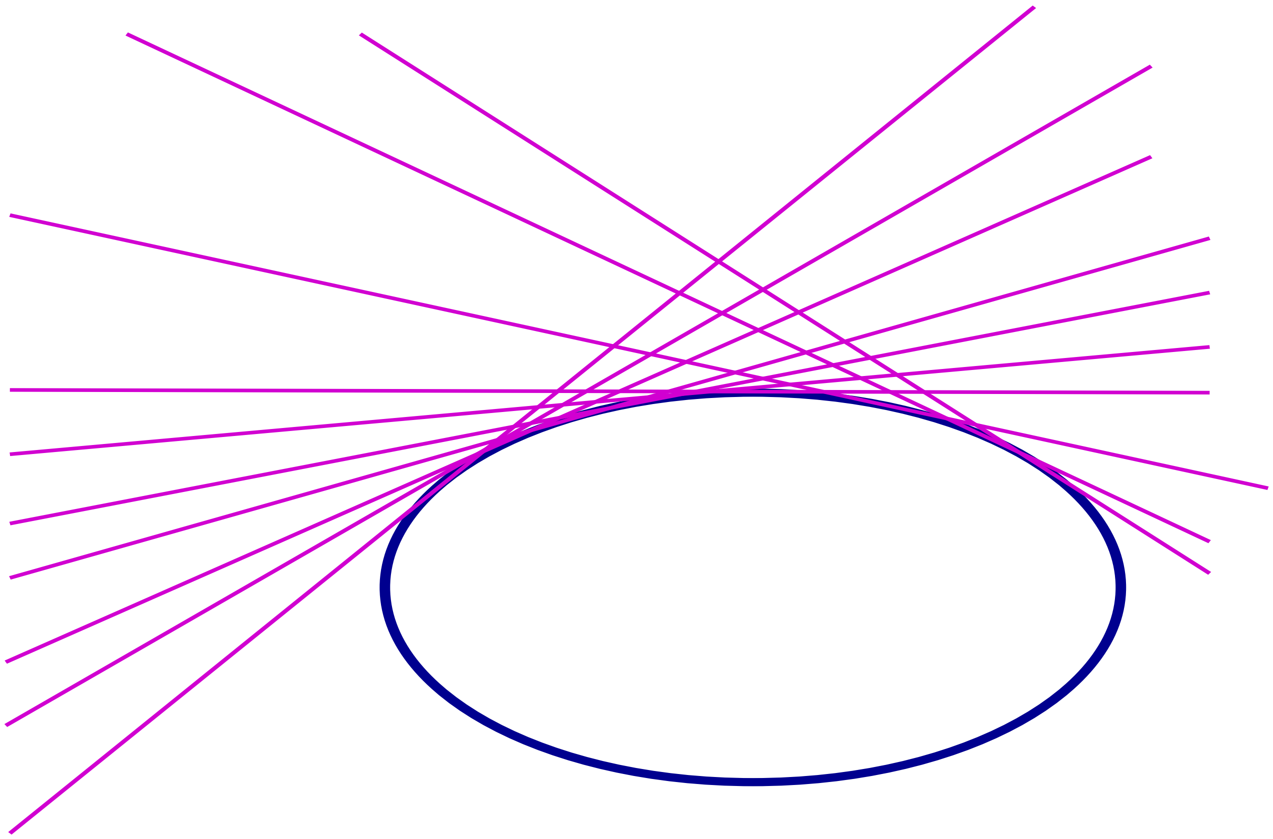
Envelope of circles, lifting





We lift the curve of parameter t to height t, getting a surface, the horizontal sections of which are the parametrized curves.

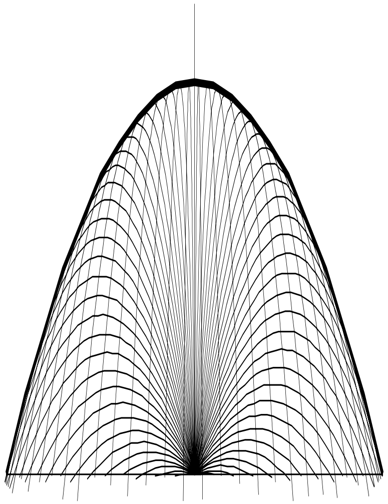
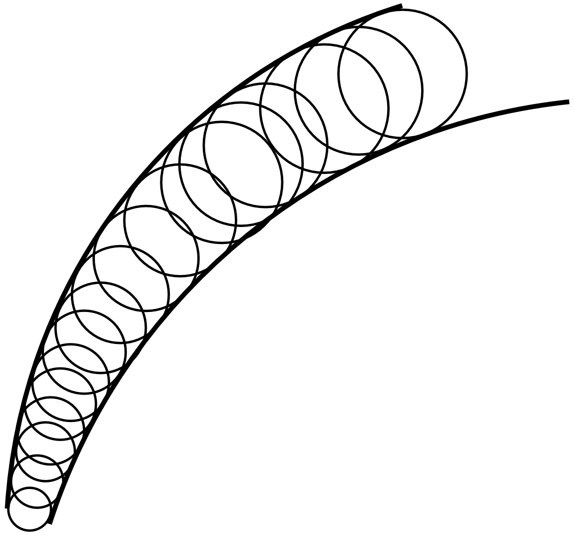
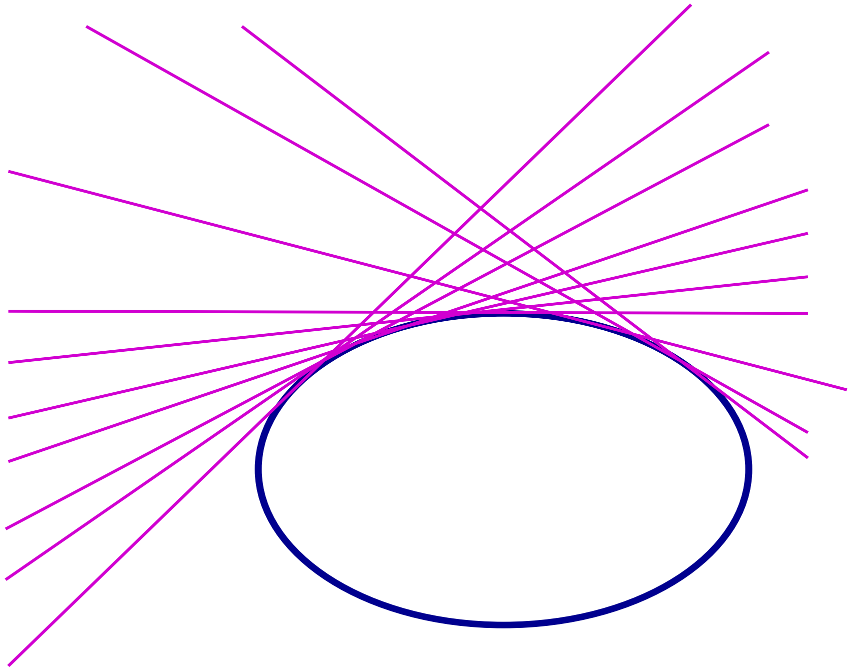
Envelope of tangents



If we have a nice curve, its tangents (may) form a 1-parameter family of straight lines and this curve may be their envelope.

Envelope covered by the curves

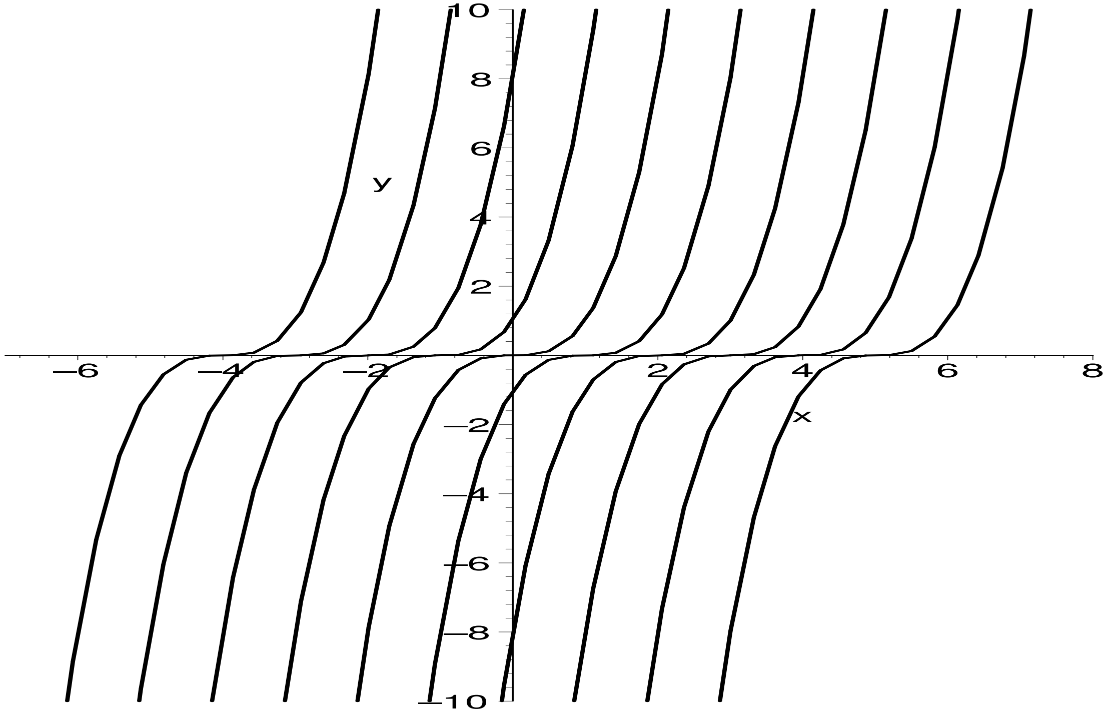
The geometric picture: We have a 1-parameter family that covers some part of the plane, does not cover some other part, and the **borderline** is the enveloping curve.

Wrong! The curves can cover their envelope

Envelope covered in the plane

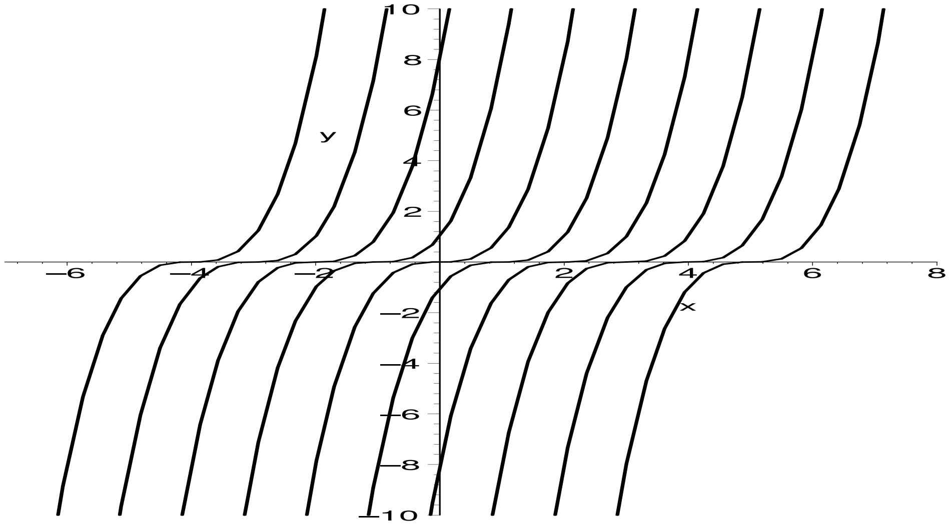
Shift the curve $y = x^3$ by t:



$$y = (x - t)^3;$$

then axis $y = 0$ will be an envelope.

Analytically:

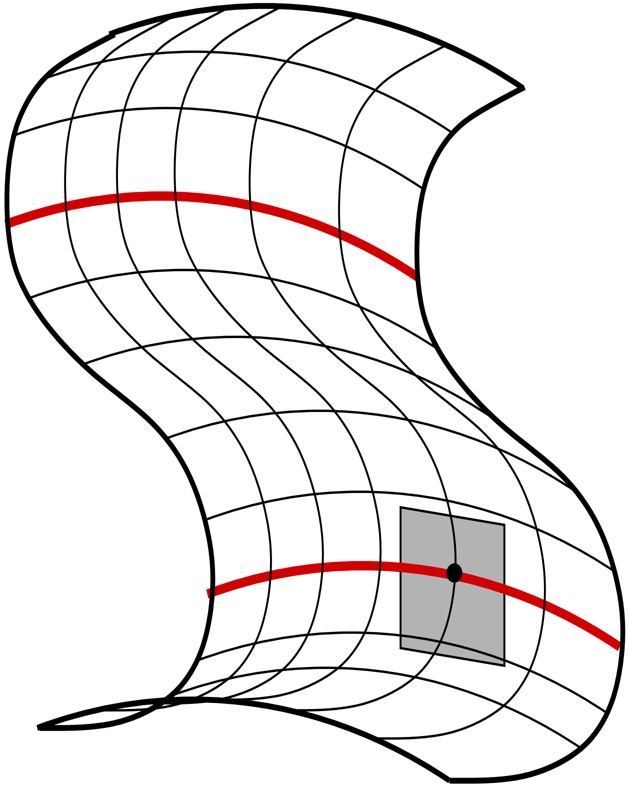


Natural form	$F(x,y,t)=0$	$F_t(x,y,t)=0$
$y=(x-t)^3$	$(x-t)^3-y=0$	$3(x-t)^2+y=0$
		$(x-t)^3+(y/3)^{3/2}=0$

$$(y/3)^{3/2}+y=0$$

then axis $y=0$ will be an envelope…

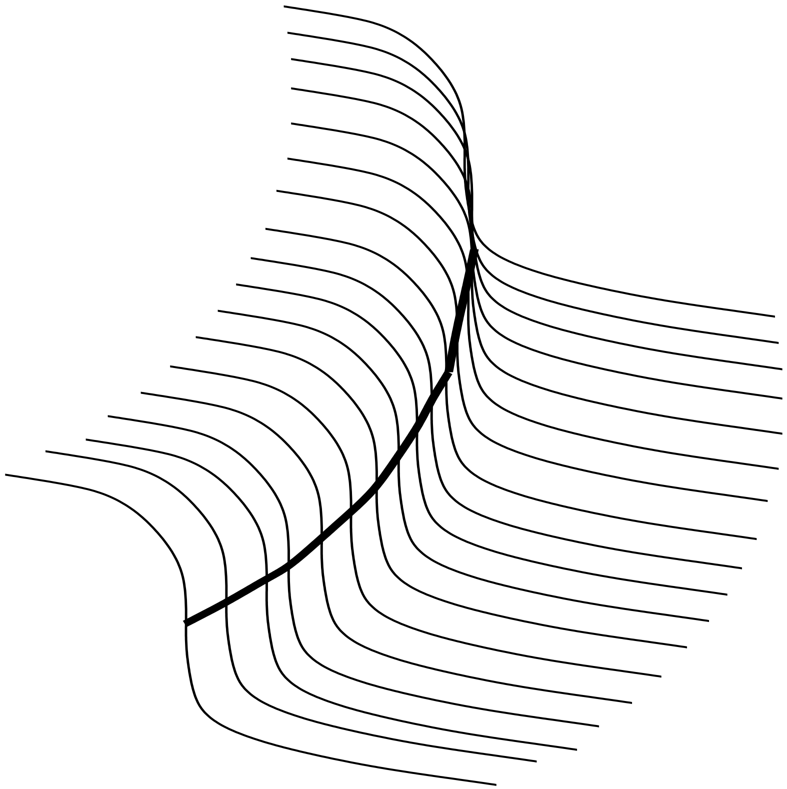
Envelope and the lifted surface

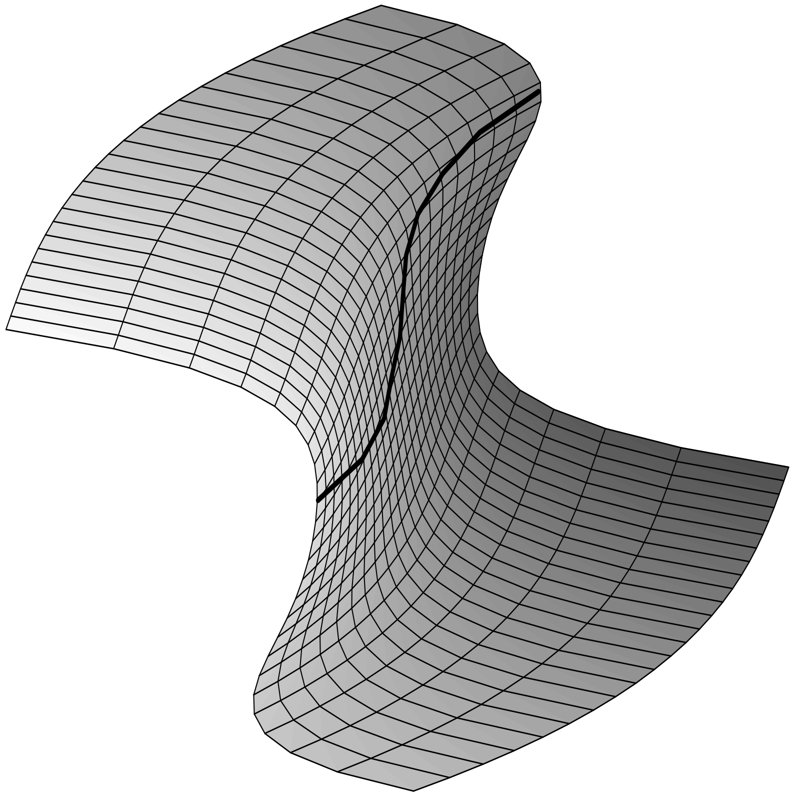


Lifting: The lifted family of curves is a surface: if the curve of parameter t goes through (x, y), then this point is lifted to (x, y, t).

In the lifted family of curves (=surface) those vertices project onto the envelope where the **tangent plane is vertical**

Envelope covered by the curves, again

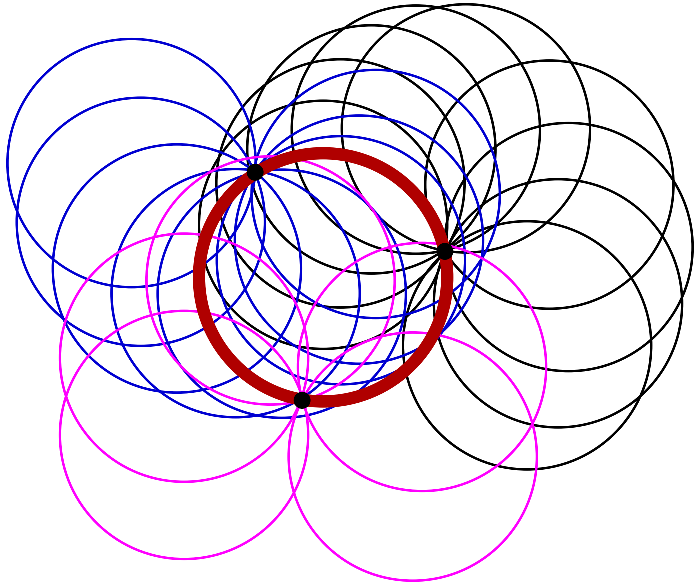


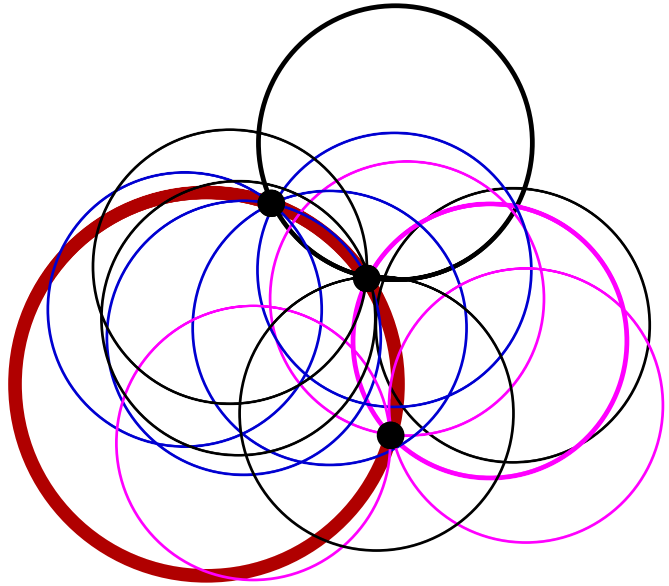


The lifted 1-parameter family yields a surface, the points of the enveloping curve come from those where the tangent planes are vertical. (Thick black curve)

Degenerate Cases

Unit circles through three points:





Here there may be a UNIT circle going through the fixed points A, B, C of the 3 families of unit circles. If we counted the crossings with multiplicity, this should have been excluded: A, B, C contribute ∞ to the number of triple points.

By the way, the points of the red unit circle are all triple points!

Concurrency function: Describes the triple points:

Consider the parametrizations and solve them: **express the parameters**.

$$F_1(x,y,t)=0 \quad \Leftrightarrow \quad t=\varphi_1(x,y)$$
$$F_2(x,y,u)=0 \quad \Leftrightarrow \quad u=\varphi_2(x,y)$$
$$F_3(x,y,v)=0 \quad \Leftrightarrow \quad v=\varphi_3(x,y)$$

Mostly two parameters define (x,y) that defines the third parameter: Three curves meet iff

$$\Psi(t,u,v)=0 \quad \text{i.e.} \quad \Psi(\varphi_1(x,y),\varphi_2(x,y),\varphi_3(x,y))\equiv 0$$

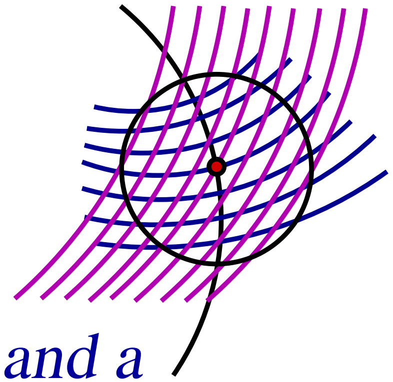
Here $\varphi_i(x,y)$ may be many-valued functions. This creates some difficulties.

We shall assume that $\Psi(.,.,.)$ is a **polynomial**.

Ψ is the **concurrency** function.

Main Theorem (slight change in the notation)

Assumptions: $\Gamma_1, \Gamma_2, \Gamma_3$ *be 1-paremeter families of curves implicitly parametrized by* F_1, F_2, F_3, *analytic on domains* $G_i \times T_i$ *and continuous on* $\mathrm{cl}(G_i \times T_i)$. *Assume that the concurrency function* $\Psi = \Psi(t_1, t_2, t_3) \in \mathcal{C}[t_1, t_2, t_3]$ *is a polynomial:*
If $t := \varphi_i(x, y)$ *is an analytic branch of the solutions of* $F_i(x, y, t) = 0$, *for* $i = 1, 2, 3$, *then*



$$(1) \qquad \Psi(\varphi_1(x, y), \varphi_2(x, y), \varphi_3(x, y)) = 0$$

(i) Γ_3 *has a partial envelope* $\mathcal{E}$;

(ii) $\mathcal{E} \subseteq G_1 \cap G_2 \cap \mathrm{cl}(G_3)$; *and there is a point* $P \in \mathcal{E}$ *and a neighbourhood* U *of it such that* Γ_1 *and* Γ_2 *both cover* U.

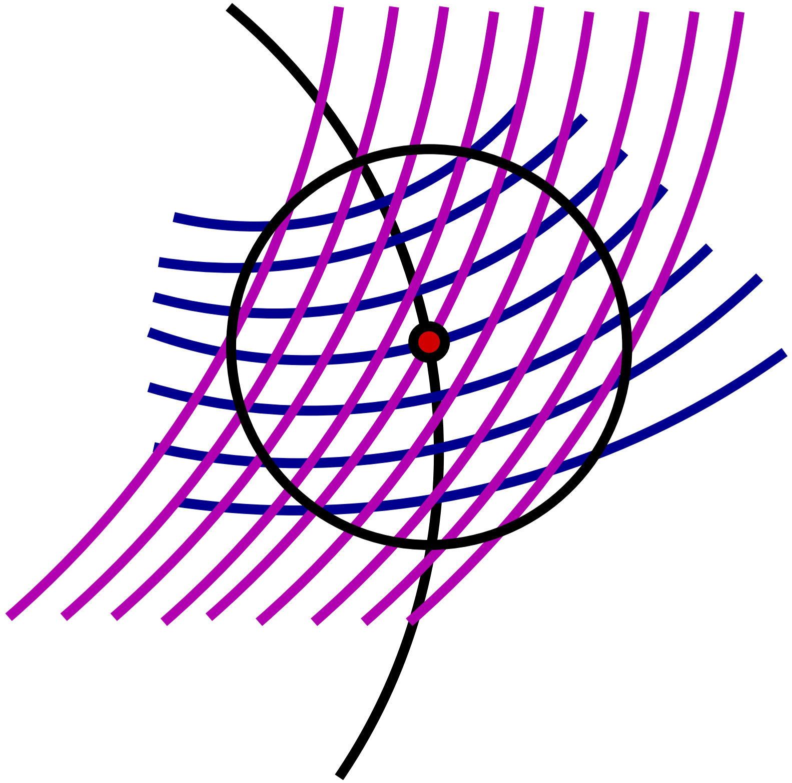
(iii) No sub–arc of $\mathcal{E}$ *is contained in any* $\gamma \in \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$.

THEN:

$$\mathcal{T}_{\Gamma_1, \Gamma_2, \Gamma_3}(n) = \mathcal{O}(n^{2-\eta}),$$

for suitable $\eta = \eta(\deg(F))$ *and* $n > n_0 = n_0(\deg(F))$.

The geometric conditions



The geometric assumption is that one of the families has a partial envelope, and the other two cross it transversally: the three tangents are distinct.

We have two versions of this: two theorems

The role of the concurrency function

Consider a grid: (a, b, c) with integer coordinates.

- A linear surface: $x + y + z = c$ may go through many grid-points.

Transform the grid, e.g.:

- $(\log a, \log b, \log c)$ with a, b, c integers. $xyz = const$ iff $\log x + \log y + \log z = const$: goes through many of the transformed grid-points.

- The opposite is also true: A nice function, going through very many grid-points should be of very simple form.

Distinguishing Circles from Straight Lines

There exist an absolute constant $\eta \in (0, 1)$ and a threshold n_0 with the following property:

Let (a_1, b_1), (a_2, b_2), (a_3, b_3) be three distinct points in the Euclidean plane and Γ_1, Γ_2, Γ_3 be three families of unit circles, such that, for each $i \leq 3$, all circles of Γ_i pass through the common point (a_i, b_i). Then

$$\mathcal{T}_{\Gamma_1,\Gamma_2,\Gamma_3}(n) = O(n^{2-\eta}),$$

provided that $n > n_0$.

Elekes-Szabó: Black Box

For any $d \in \mathbb{Z}^+$ there exist an $\eta = \eta(d) \in (0,1)$, $k = k(d)$ and $n_0 = n_0(d)$ such that if $V \subset \mathcal{C}^3$ is a two dimensional algebraic surface of degree $\leq d$ and for infinitely many n there exist $T_1, T_2, T_3 \subset \mathcal{C}$ such that $|T_1| = |T_2| = |T_3| = n$ and

$$|V \cap (T_1 \times T_2 \times T_3)| \geq n^{2-\eta};$$

then either V is cylindric or there is a point $P \in V$ and a neighbourhood $\mathcal{U}$ of P where the surface can be transformed into

$$x + y + t = 0,$$

by a curvilinear transformation.

More explanation is needed here: ...

Pseudo-grids and surfaces (meaning of ...)

If a nice surface contains cn^2 generalized lattice points, then it **must be very special**

After **curvilinear rescaling** we get

$$t + u + v = 0.$$

Multivalued functions, branches

We use multivalued functions, the surface V can be described locally by

$$0 \in \varphi_1(t_1) + \varphi_2(t_2) + \varphi_3(t_3)$$

that corresponds to

$$\varphi_1(t_1) + \varphi_2(t_2) + \varphi_3(t_3) = 0,$$

Significant “jump”: for a fixed $\eta > 0$ (depending on the degree of V, either V has only $O(n^{2-\eta})$ generalized grid ponts, or it has at least cn^2.

Remarks

Sufficient condition for three one–parameter families of curves (or for three copies of a single family) to have few triple intersections.

How far below quadratic should it be? Since we have no reasonable estimate for $\eta > 0$, nothing is known about the exact order of magnitude.

It may well be that the number of triple points is at most $n^{1+\varepsilon}$, for any $\varepsilon > 0$.

Historical remarks and examples

Earlier results for straight lines

Studying the incidence structures of points and straight lines (more generally, of points and certain curves) has been one of the fundamental tasks of Combinatorial Geometry for long.

140 years ago Sylvester: famous “Orchard Problem” which, in a dual form, asks for arranging n straight lines in the Euclidean plane so that **the number of triple points be maximized. Sylvester** showed that if $\mathcal{L}$ is the family of all straight lines then $\mathcal{T}_{\mathcal{L}}(n) = n^2/6 + \mathcal{O}(n)$.

Later on **Burr, Grünbaum and Sloan** slightly improved his lower bound.

Earlier results on unit circles

An "orchard–like" problem was posed by Erdős:

arrange n unit circles in the Euclidean plane so that the number of triple points be maximized.

Denoting the family of all unit circles by $\mathcal{U}$, an upper for the above problem by $\mathcal{T}_{\mathcal{U}}(n)$. Now $\mathcal{T}_{\mathcal{U}}(n) \leq n(n-1)$ is obvious (since, as before, already the number of *pairwise* intersections obeys this bound). Also, a lower bound of $\mathcal{T}_{\mathcal{U}}(n) \geq cn^{3/2}$ was proved by Elekes. The gap between these two estimates is still wide open.

Earlier results on unit circles

Also from another point of view, unit circles play a special role in Combinatorial Geometry. One of the most challenging unsolved problems is

Conjecture (Erdős). *For any* $\varepsilon > 0$ *there is an* n_0 *such that for* $n > n_0$ *the maximum possible number* $u(n)$ *of unit distances between* n *points in* $\mathbb{R}^2$ *is at most* $n^{1+\varepsilon}$.

Again, Happy Birthday, Tom!

Happy birthday!