LOGICAL AXIOMATIZATIONS OF SPACE-TIME. SAMPLES FROM THE LITERATURE

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Abstract We study relativity theory as a theory in the sense of mathematical logic. We use first-order logic (FOL) as a framework to do so. We aim at an "analysis of the logical structure of relativity theories". First we build up (the kinematics of) special relativity in FOL, then analyze it, and then we experiment with generalizations in the direction of general relativity. The present paper gives samples from an ongoing broader research project which in turn is part of a research direction going back to Reichenbach and others in the 1920's. We also try to give some perspective on the literature related in a broader sense. In the perspective of the present work, axiomatization is not a final goal. Axiomatization is only a first step, a tool. The goal is something like a *conceptual analysis of relativity* in the framework of logic.

In section 1 we recall a complete FOL-axiomatization **Specrel** of special relativity from [5],[31]. In section 2 we answer questions from papers by Ax and Mundy concerning the logical status of faster than light motion (FTL) in relativity. We claim that already very small/weak fragments of **Specrel** prove "*No FTL*". In section 3 we give a sketchy outlook for the possibility of generalizing **Specrel** to theories permitting accelerated observers (gravity). In section 4 we continue generalizing **Specrel** in the direction of general relativity by localizing it, i.e. by replacing it with a version still in first-order logic but now local (in the sense of general relativity theory). In section 5 we give samples from the broader literature.

Keywords: logic, first-order logic, axiomatization, completeness theorem, foundation of space-time, special and general relativity, faster-than-light motion, reverse geometry, reverse spacetime theory, Alexandrov-Zeeman type theorem, accelerated observers, local theory

Introduction

The interplay between logic and relativity theory goes back to around 1920 and has been playing a non-negligible role in works of researchers like Reichenbach, Carnap, Suppes, Ax, Szekeres, Malament, Walker, and of many other contemporaries. For example, the logical theory of definability can find its roots in this interplay, cf. Reichenbach [41](1924). On the other side, definability has always been an important topic in the foundation of relativity from Reichenbach (1924) to e.g. Malament (1977) [35], Friedman (1983), [16] (1991), and Hogarth (2003) [26].

The present paper intends to give samples from the area called *analysis of the logical structure of relativity theories.* The first step in this analysis is building up relativity theory as a theory in the sense of first-order logic (FOL). The reason why we chose FOL and not e.g. second-order logic is presented in detail in [5, App.1] as well as in Ax [7], Pambuccian [40], but the reasons in Väänänen [54], Ferreirós [15], or Woleński [58] also apply.[1] Axiomatizations of special relativity have been extensively studied in the literature, cf. section 5. These works usually stop with a kind of completeness theorem for their axiomatizations. What we call the analysis of the logical structure of relativity theory begins with proving such a completeness theorem but the real work comes afterwards, during which one often concludes that we have to change the axioms. Very roughly, one could phrase this as "we start off where the others stopped (namely, at completeness)". In the present work, especially in section 2, we try to illustrate what we understand by this kind of analysis of logical structure.

In sections 1 and 2 we recall (from [5],[31]) and study a, basically, complete FOL-axiomatization **Specrel** of special relativity. In sections 3, 4 we give an outlook on generalizing the logic based approach towards general relativity. We will make "two steps" towards general relativity: In section 3 we extend **Specrel** by permitting accelerated observers and this way we become able to study some aspects of gravity via Einstein's equivalence principle. The other step is in section 4 where we make our theory local in the sense of general relativity. We call this process localization of a theory and we note that it is related to the process of relativization which already proved successful in algebraic logic and in the area known as "Logic, Language and Information" (LLI), cf. e.g. [6]. We do all this in the framework of first-order logic. Our intention is to use these "two steps" towards general relativity together, i.e. to combine the theories obtained by the methods of sections 3 and 4.

Axiomatization of a subject matter (like e.g. geometry) usually evolves through 3 stages. The first to appear is an axiomatization without formal logic, followed by an axiomatization in the framework of formal

[1] Being in the framework of FOL is the same as being *elementary*, according to the standard terminology, cf. e.g. Tarski [53].

(but perhaps higher-order) logic, and finally by an axiomatization in the framework of first-order logic (which might be many-sorted). In the case of geometry, Euclid gave the first kind of axiomatization, Hilbert gave the second, and finally Tarski and his school developed the third, cf. e.g. [53], [40]. In the case of set theory, it was Cantor, Russell-Zermelo, modern set theory. The history of the "nonlogical", "higher-order logic", "first-order logic" stages of evolution of axiomatizations for some other theories (e.g. Peano's Arithmetic) can be found e.g. in Ferreirós [15]. This is a natural temporal sequence of development, since each one of these stages "prepares the ground" for the next one. For the purposes of conceptual analysis the first-order-logic-based axiomatizations are the most suitable, e.g. because FOL forces us to be explicit about many things.

As we will see in section 5, the *interplay between logic and relativity* has been studied in the literature. Turning to axiomatizations as special parts of this interplay, a large part of the axiomatizations of relativity in the literature are of the first stage (nonlogical) and concern special relativity. There is a growing number of works which are in the framework of formal logic, e.g. the book Schutz [46] (second stage). Some are already in the framework of FOL; these will be further reflected on in section 5 from the point of view of e.g. amenability for conceptual analysis or other purposes of insight-seeking. Motivation for the research direction surveyed/reported here is nicely summarized in Ax [7], Suppes [48]; cf. also the introduction of [5]. Harvey Friedman's [17], [18] present a rather convincing general perspective (and motivation) for the kind of work reported here.[2]

1. An axiomatization of special relativity, in FOL

In this paper we deal with the kinematics of relativity only, i.e. we deal with motion of *bodies* (or *test-particles*). The motivation for our choice of vocabulary (for special relativity) is summarized as follows. We will represent motion as changing spatial location in time. To do so, we will have reference-frames for co-ordinatizing events and, for simplicity, we will associate reference-frames with special bodies which we will call *observers*. We visualize an observer-as-a-body as "sitting" in the origin of its reference frame, or equivalently, "living" on the time-axis of the reference frame. There will be another special kind of bodies which we

[2]In passing we mention that Etesi-Németi [13], Hogarth [27] represent further kinds of connection between logic and relativity not discussed here.

will call *photons*. For co-ordinatizing events we will use an arbitrary *ordered field* in place of the field of the real numbers. Thus the elements of this field will be the "*quantities*" which we will use for marking time and space.

Let us fix a natural number $n > 1$. n will be the number of space-time dimensions. In most works $n = 4$, i.e. one has 3 space-dimensions and one time-dimension.[3]

Motivated by the above, our language contains the following symbols:

unary relation symbols $\mathsf{B}, \mathsf{Ob}, \mathsf{Ph}, \mathsf{F}$ (for bodies, observers, photons, and quantities, i.e. elements of the field, respectively),

binary function symbols $+, -, \cdot, /$, constants $0, 1$ and a binary relation symbol $<$ (for the field-operations and ordering on F),

a $2+n$-ary relation symbol W (for co-ordinatizing events, i.e. for the world-view relation).

We will read "$\mathsf{B}(x), \mathsf{Ob}(x), \mathsf{Ph}(x), \mathsf{F}(x)$" as "$x$ is a body", "x is an observer", "x is a photon", "x is a field-element", and we will read "$\mathsf{W}(x, y, z_1, z_2, \dots, z_n)$" as "observer x sees (or observes) the body y at time z_1 at location $(z_2, \dots z_n)$". This "seeing" or "observing" has nothing to do with seeing via photons or observing via experiments, it simply means that, according to x's co-ordinate system or reference frame, y is present at co-ordinates $(z_1, \dots, z_n)$.

The above, together with statements of the form $x = y$, $x < y$ are the so-called *atomic formulas* of our first-order language, where $x, y, \dots, z_n$ can be arbitrary variables or terms built up from variables by using the field-operations $+, -, \cdot, /, 0, 1$. The *formulas* of the first-order language are built up from these atomic formulas by using the logical connectives *not* ($\neg$), *and* ($\wedge$), *or* ($\vee$), *implies* ($\rightarrow$), *if-and-only-if* ($\leftrightarrow$) and the quantifiers *exists* ($\exists$) and *for all* ($\forall$).

Usually we will use the variables m, k to denote observers, b to denote bodies, ph to denote photons and $p_1, \dots, q_1, \dots$ to denote field-elements. We will write p and q in place of $p_1, \dots, p_n$ and $q_1, \dots, q_n$, e.g. we will write $\mathsf{W}(m, b, p)$ in place of $\mathsf{W}(m, b, p_1, \dots, p_n)$, and we will write $\forall p$ in place of $\forall p_1, \dots, p_n$ etc.

The models of this language are of the form $\mathsf{M} = \langle U, \mathsf{B}, \mathsf{Ob}, \mathsf{Ph}, \mathsf{F}, \dots, \mathsf{W}\rangle$ where U is a nonempty set, $\mathsf{B}, \mathsf{Ob}, \mathsf{Ph}, \mathsf{F}$ are unary relations on U, etc.

[3]Recent generalizations of general relativity in the literature (e.g. M-theory) indicate that it might be useful to leave n as a variable.

A unary relation on U is just a subset of U, so we will use B, Ob etc as sets as well, e.g. we will write $x \in \mathsf{B}$ in place of $\mathsf{B}(x)$. In our investigations, we will always assume that $\mathsf{Ob}, \mathsf{Ph} \subseteq \mathsf{B}$, and that $U = \mathsf{B} \cup \mathsf{F}$ is the universe of M. Therefore we will write our models in the form $\mathsf{M} = \langle \mathsf{B}, \mathsf{Ob}, \mathsf{Ph}, \underline{\mathsf{F}}, \mathsf{W} \rangle$ where $\underline{\mathsf{F}} = \langle \mathsf{F}, +, \cdot, -, /, 0, 1, < \rangle$ will be an ordered field with universe F, and $\mathsf{B} \cup \mathsf{F}$ plays the role of the "universe" of M.

Having fixed our language, we now turn to formulating an axiom system for special relativity in this language.

We will formulate each axiom on three levels. First we give a very intuitive formulation, then we give a precise formalization using notions that will be useful later as well (like life-line), and finally, for completeness, we give a concrete first-order formula without using the introduced notions and abbreviations, at the beginning, at least. Readers familiar with first-order logic can read the first-order formulas right away.

${}^n\mathsf{F}$ denotes the set of all n-tuples of elements of F. If a is an n-tuple, then we will assume that $a = \langle a_1, \ldots, a_n \rangle$, i.e. a_i denotes the i-th member of the n-tuple a (for $0 < i \leq n$). We will use the *vector-space* structure of ${}^n\mathsf{F}$. I.e. if $p, q \in {}^n\mathsf{F}$ and $\lambda \in \mathsf{F}$, then $p + q, p - q, \lambda p \in {}^n\mathsf{F}$, and $\overline{0} = \langle 0, \ldots, 0 \rangle$ is the *origin.* The *slope* of a vector $p \in {}^n\mathsf{F}$ is defined as $\mathsf{slope}(p) = (p_2^2 + \ldots + p_n^2)/p_1^2$ if $p_1 \neq 0$, and $\mathsf{slope}(p) = \infty$ otherwise, where $\infty \notin \mathsf{F}$ is fixed, as usual, to denote a kind of formal infinity.

Let $q, v \in {}^n\mathsf{F}, v \neq \overline{0}$. The (straight) *line* going through q and with *squared speed* [4] or *slope* $(v_2^2 + \ldots + v_n^2)/v_1^2$ and in the *spatial direction* $\langle v_2, \ldots, v_n \rangle$ is

$\{q + \lambda v : \lambda \in \mathsf{F}\}$. The set of straight lines is then

$\mathsf{Lines} \stackrel{d}{=} \{\{q + \lambda v : \lambda \in \mathsf{F}\} : q, v \in {}^n\mathsf{F}, v \neq \overline{0}\}$. If $\ell \in \mathsf{Lines}$, then

$\mathsf{slope}(\ell) \stackrel{d}{=} \mathsf{slope}(p - q)$ for some (and then for all) $p, q \in \ell, p \neq q$.

The *life-line*, or *<u>tr</u>ace* of a body b in observer m's world-view, or as seen by m, is the set of co-ordinate points at which m sees b, and the set of bodies m sees at a given co-ordinate point p is the *event* happening for m at p:

$\mathsf{tr}_m(b) \stackrel{d}{=} \{p \in {}^n\mathsf{F} : \mathsf{W}(m, b, p)\}$ and

$\mathsf{ev}_m(p) \stackrel{d}{=} \{b \in B : \mathsf{W}(m, b, p)\}$.

[4]For technical reasons, we use the square of the speed instead of speed, throughout.

The *speed* of a body b as seen by m is

$\mathsf{speed}_m(b) \stackrel{d}{=} \mathsf{slope}(\mathsf{tr}_m(b))$, this is meaningful when $\mathsf{tr}_m(b) \in \mathsf{Lines}$.

We are ready now for formulating the axioms on all three levels.

AxSelf Each observer sees himself standing still at the origin, i.e. the life-line of m in his own world-view is the *time-axis*

$\mathsf{tr}_m(m) = \bar{t} = \{\langle t, 0, \ldots, 0\rangle : t \in \mathsf{F}\}$, if $m \in \mathsf{Ob}$. A first-order formula expressing this without abbreviations is

$$(\forall m, p)\Big((\mathsf{Ob}(m) \wedge \mathsf{F}(p_1)) \rightarrow [\mathsf{W}(m, m, p) \leftrightarrow (p_2 = 0 \wedge \ldots \wedge p_n = 0)]\Big).$$

AxPh The photon-traces are exactly the lines with slope 1 (in each observer's world-view). In other words, all photons move with speed 1, and it is possible to send out a photon with speed 1 at each point and in each direction,

$\{\mathsf{tr}_m(\mathsf{ph}) : \mathsf{ph} \in \mathsf{Ph}\} = \{\ell \in \mathsf{Lines} : \mathsf{slope}(\ell) = 1\}$ for all $m \in \mathsf{Ob}$. A first-order formula expressing this without abbreviations is

$$(\forall m, p, q)(\mathsf{Ob}(m) \rightarrow [(\exists \mathsf{ph})(\mathsf{Ph}(\mathsf{ph}) \wedge \mathsf{W}(m, \mathsf{ph}, p) \wedge \mathsf{W}(m, \mathsf{ph}, q)) \leftrightarrow (p_1 - q_1)^2 = (p_2 - q_2)^2 + \ldots + (p_n - q_n)^2]).$$

AxEvent All observers see the same events,

$\{\mathsf{ev}_m(p) : p \in {}^n\mathsf{F}\} = \{\mathsf{ev}_k(p) : p \in {}^n\mathsf{F}\}$ for all $m, k \in \mathsf{Ob}$. A first-order formula expressing this without abbreviations is

$$(\forall m, k, p)(\exists q)\Big((\mathsf{Ob}(m) \wedge \mathsf{Ob}(k)) \rightarrow (\forall b)[\mathsf{W}(m, b, p) \leftrightarrow \mathsf{W}(k, b, q)]\Big).$$

AxField The usual first-order axioms saying that $\langle \mathsf{F}, +, -, \cdot, /, 0, 1, <\rangle$ is a *Euclidean* ordered field, i.e. an ordered field in which positive elements have square-roots; together with the formula expressing $\mathsf{Ob}, \mathsf{Ph} \subseteq \mathsf{B}$ and $\mathsf{W} \subseteq \mathsf{Ob} \times \mathsf{B} \times {}^n\mathsf{F}$.

$$\mathbf{Specrel}_0 \stackrel{d}{=} \{\mathbf{AxSelf}, \mathbf{AxPh}, \mathbf{AxEvent}, \mathsf{AxField}\}.$$

Since, in some sense, AxField is only an "auxiliary" axiom about the "mathematical frame" of our reasoning, the heart of $\mathbf{Specrel}_0$ consists of 3 very natural axioms, **AxSelf**, **AxPh**, **AxEvent**. The reader is invited to check that these are really intuitively convincing, natural and simple assumptions.

From these four axioms already one can prove the most characteristic predictions of special relativity theory. What the average layperson usually knows about relativity is that "moving clocks slow down", "moving spaceships shrink", and "moving pairs of clocks get out of synchronism". We call these the *paradigmatic effects* of special relativity. All these can be proven from the above four axioms, in some form. E.g. one can prove that "if m, k are any two observers not at rest relative to each other, then one of m, k will "see" or "think" that the clock of the other runs slow". Careful analysis, statement, and proofs (from our axioms) of the various paradigmatic effects of special relativity can be found in [5, §§2.5, 2.6, 2.8, 4.8] and in [31].

As an example here we formulate that time cannot be taken as absolute[5], i.e. it is impossible to construct a model for the above four axioms in which all observers agree about what events are happening simultaneously (assuming not all the observers are at rest w.r.t. each other). From now on we do not spell out the concrete, abbreviation-free first-order formulas, it will always be straightforward to construct them (if someone wanted to see them). Let

NoAbstime $(\exists m, k, p, q, p', q')[p_1 = q_1 \wedge p'_1 \neq q'_1 \wedge \mathsf{ev}_m(p) = \mathsf{ev}_k(p') \wedge \mathsf{ev}_m(q) = \mathsf{ev}_k(q')]$.

Intuitively, events $\mathsf{ev}_m(p)$ and $\mathsf{ev}_m(q)$ are simultaneous for m but not simultaneous for k.

NotAllrest
$(\exists m, k, p, q)[\mathsf{W}(m, k, p) \wedge \mathsf{W}(m, k, q) \wedge (p_2 \neq q_2 \vee \ldots \vee p_n \neq q_n) \wedge \mathsf{Ob}(k)]$.

Intuitively, k is not at rest relative to m.

If Σ is a set of formulas and M a model, then $\mathsf{M} \models \Sigma$ denotes that all formulas in Σ are true in the model M, in this case we say that M is *a model of* Σ. If φ is a formula, then $\Sigma \models \varphi$ denotes that φ is true in all models of Σ. In this case we say that φ *follows from* (or, *is a semantical*

[5]In the more general framework towards which we will be working in section 3, stronger theorems will be provable about time not being absolute (cf. e.g. Gödel's rotating universes [21], [31, Fig.135, p.366]). However, making this precise has to be postponed.

consequence of) Σ. By Gödel's completeness theorem, this implies that φ is *provable* from Σ in the usual, syntactical sense of first-order logic's proof theory.

THEOREM 1.1 **Specrel**$_0$ $\models$ (**NotAllrest** $\rightarrow$ **NoAbstime**), *assuming* $n > 2$.

Theorem 1.1 follows from Theorem 1.2 way below.

Adding the next axiom **AxSym** to **Specrel**$_0$ will imply all the paradigmatic effects of relativity in their strongest form (and even the so-called Twin Paradox in a form), cf. Thm.1.2(ii). The *world-view transformation* f_{mk} between two observers m, k is defined as

$$\mathsf{f}_{mk} \stackrel{d}{=} \{\langle p, q\rangle : \mathsf{ev}_m(p) = \mathsf{ev}_k(q), \mathsf{ev}_k(q) \neq \emptyset \text{ and } p, q \in {}^n\mathsf{F}\} \ .$$

From the axioms so far we can see that the world-view transformations play a central role in relativity. From our previous axioms it follows that f_{mk} is a transformation of ${}^n\mathsf{F}$ (and not only an arbitrary binary relation) if m, k are observers.[6] Therefore we will use f_{mk} as a function. Then $\mathsf{f}_{mk}(p)$ is the "place" where k sees the same event that m sees at p, i.e.

$$\mathsf{ev}_m(p) = \mathsf{ev}_k(\mathsf{f}_{mk}(p)) \ .$$

Let $p, q \in {}^n\mathsf{F}$. Then $p_1 - q_1$ is the time passed between the events $\mathsf{ev}_m(p)$ and $\mathsf{ev}_m(q)$ as seen by m and $\mathsf{f}_{mk}(p)_1 - \mathsf{f}_{mk}(q)_1$ is the time passed between the same two events as seen by k. Hence $\|(\mathsf{f}_{mk}(p)_1 - \mathsf{f}_{mk}(q)_1)/(p_1 - q_1)\|$ is the rate with which k's clock runs slow as seen by m. Here, $\|a\|$ denotes the *absolute value* of a when $a \in \mathsf{F}$, i.e. $\|a\| \in \{a, -a\}$ and $\|a\| \geq 0$.

AxSym All observers see each other's clocks run slow to the same extent,

$$\|\mathsf{f}_{mk}(p)_1 - \mathsf{f}_{mk}(q)_1\| = \|\mathsf{f}_{km}(p)_1 - \mathsf{f}_{km}(q)_1\|, \text{ when } m, k \in \mathsf{Ob} \text{ and } p, q \in \bar{t}.$$

$$\textbf{Specrel} \stackrel{d}{=} \textbf{Specrel}_0 \cup \{\textbf{AxSym}\} = \{\textbf{AxSelf}, \textbf{AxPh}, \textbf{AxEvent}, \textbf{AxSym}, \mathsf{AxField}\}.$$

The (square of the) so-called *Minkowski-length* of $p \in {}^n\mathsf{F}$ is defined as

[6] This is a typical example of a property of special relativity which will be relaxed in the process of localization (towards general relativity) in sections 3,4. Namely, the axioms of our local theories in sections 3,4 will not imply that the function f_{mk} is everywhere defined in ${}^n\mathsf{F}$. This is an essential generalization towards general relativity.

$$\mu(p) \stackrel{d}{=} p_1^2 - p_2^2 - \ldots - p_n^2, \quad \text{while} \quad |p| \stackrel{d}{=} p_1^2 + p_2^2 + \ldots + p_n^2$$

denotes the (square of the) usual *Euclidean length.* Let $f : {}^n\mathsf{F} \to {}^n\mathsf{F}$ be a mapping. Recall from the literature that f is called an *affine* transformation if it is the composition of a bijective linear transformation with a translation. f is called a *Poincaré-transformation* (or an *inhomogeneous Lorentz-transformation*) if f is an affine transformation which preserves Minkowski-distance, i.e. if $\mu(f(p) - f(q)) = \mu(p - q)$ for all $p, q \in {}^n\mathsf{F}$. [7] f is called a *dilation* if there is a positive $\delta \in \mathsf{F}$ such that $f(p) = \delta p$ for all $p \in {}^n\mathsf{F}$ and f is called *field-automorphism-induced* if there is an automorphism π of the field $\langle \mathsf{F}, +, \cdot \rangle$ such that $f(p) = \langle \pi p_1, \ldots, \pi p_n \rangle$ for all $p \in {}^n\mathsf{F}$.

The following is proved in [5, 2.9.4, 2.9.5] and in [31, 2.9.4–2.9.7].

THEOREM 1.2 *Let $n > 2$, let M be a model of our language and let m, k be observers in M. Then (i)-(ii) below hold.*

*(i) f_{mk} is a Poincaré-transformation composed with a dilation and a field-automorphism-induced mapping, if $\mathsf{M} \models$ **Specrel**$_0$.*

*(ii) f_{mk} is a Poincaré-transformation, if $\mathsf{M} \models$ **Specrel**.*

Assume $n > 2$. Theorem 1.2 (i) and (ii) above are best possible in the sense that e.g. for every Poincaré-transformation f over an arbitrary Euclidean ordered field there are a model $\mathsf{M} \models$ **Specrel** and observers m, k in M such that the world-view transformation f_{mk} between m's and k's world-views in M is f, see [5, 2.9.4(iii), 2.9.5(iii)]. Similarly for (i). Hence, Thm.1.2 can be refined to a pair of completeness theorems, cf. [5, 3.6.13, p.271].

It follows from Thm.1.2 that the paradigmatic effects hold in **Specrel** in their strongest form, e.g. if m and k are observers not in rest w.r.t. each other, then both will "think" that the clock of the other runs slow. **Specrel** also implies the "inertial approximation" of the Twin Paradox, see e.g. [5, 2.8.18], and [50]. For the Twin Paradox see also section 3 herein.

[7] A concrete description of Poincaré-transformations in Euclidean fields is known, as follows. f is called an *isometry* if it preserves Euclidean length, and f is called a *Newtonian* transformation (or sometimes "trivial") if it is an isometry which is the identity on the time axis $\bar{t}$, composed with a translation. (Indeed, Newtonian, or Galilean, re-coordinatizations are such.) f is called a *Lorentz-boost* if there is $0 \leq v < 1$ such that $f(p_1, \ldots, p_n) = \langle (p_1 - vp_2)/\sqrt{1-v^2}, (p_2 - vp_1)/\sqrt{1-v^2}, p_3, \ldots, p_n \rangle$ for all $p \in {}^n\mathsf{F}$. Now, it is known that f is a Poincaré-transformation iff it is a composition of a Newtonian transformation, a Lorentz-boost, and another Newtonian transformation (cf. e.g. [5, §2.9]).

By Theorem 1.2(ii), **Specrel** implies that the f_{mk}'s are affine transformations (i.e. linear + translation). In models of $\mathbf{Specrel}_0$, the world-view-transformations are not necessarily affine transformations, but (by Thm.1.2(i)) they are still line-preserving bijections (or, in other words, *collineations*), i.e. the image of any line under a world-view transformation is a line again. That the f_{mk}'s are collineations follows from the axioms in $\mathbf{Specrel}_0$ by a suitable generalization of the following theorem. Let $\Re$ denote the ordered field of real numbers, also called the *real line*.

THEOREM 1.3 (Alexandrov and Zeeman) *Let $n > 2$. If a bijection $f : {}^n\Re \to {}^n\Re$ maps lines of slope 1 onto lines of slope 1, then f maps any line onto a line, i.e. f is a collineation.*

Theorem 1.3 is very useful and has many strengthened versions and variants in the literature. For a survey of these see Guts [23] and Lester [29]. In particular, Theorem 1.3 is true if we replace $\Re$ with any Euclidean ordered field in it (cf. e.g. [55], [56]). (Thm.s 4.1,4.2 in section 4 herein are also Alexandrov-Zeeman type theorems.)

The condition $n > 2$ is necessary in Thm.1.2, because the Alexandrov-Zeeman theorem is not true for $n = 2$ and for this same reason, Theorem 1.2 is not true, either, for $n = 2$. However, a version of Theorem 1.2 becomes true for $n = 2$ as well, if we add the following two axioms to $\mathbf{Specrel}_0$.

It has been a basic principle in physics that if no external force is acting on a body b (i.e. if it is *inertial*) then it changes neither the speed nor the direction of its movement, and this means that b's life-line is a straight line; this is expressed by the axiom

AxLine The life-line of an observer is a straight line as seen by any other observer,

$\mathsf{tr}_m(k) \in \mathsf{Lines}$ if $m, k \in \mathsf{Ob}$.

AxOb It is possible to move with any given speed less than 1, everywhere and in every direction, i.e. for all $\ell \in \mathsf{Lines}$ and $m \in \mathsf{Ob}$

$\mathsf{slope}(\ell) < 1 \to (\exists k \in \mathsf{Ob})\ell = \mathsf{tr}_m(k)$.

Let $\rho : {}^2\mathsf{F} \longrightarrow {}^2\mathsf{F}$ denote "*reflection to the $x = y$-line*", i.e. $\rho(x, y) = \langle y, x\rangle$ for all $x, y \in \mathsf{F}$. Notice that ρ is a linear transformation but it does not preserve the Minkowski-distance (since $\mu(\rho(x, y)) = -\mu(x, y)$).

THEOREM 1.4 *Let* $n = 2$*, let* M *be a model of our language and let* m, k *be observers in* M*. Then (i)-(ii) below hold.*

(i) f_{mk} *is a Poincaré-transformation composed with a dilation and a field-automorphism-induced mapping, and composed perhaps with the reflection* ρ*, if* $\mathsf{M} \models \mathbf{Specrel}_0 \cup \{\mathbf{AxLine}, \mathbf{AxOb}\}$*.*

(ii) f_{mk} *is a Poincaré-transformation composed perhaps with the reflection* ρ*, if* $\mathsf{M} \models \mathbf{Specrel} \cup \{\mathbf{AxLine}, \mathbf{AxOb}\}$*.*

Theorems 1.2, 1.4 say that $\mathbf{Specrel}_0$ and **Specrel** are adequate axiomatizations for the "metric-free" and "metric" parts of special relativity, respectively[8], because the decisive part of the completeness theorem for special relativity is the description of the world-view transformations. In [5, Thm.3.8.14, p.301], [3, Thm.4, p.15] and [4, §§6,7] these theorems are extended to complete first-order axiomatizations of the standard models of special relativity (by adding some more, "book-keeping", axioms, which are called "book-keeping" axioms because they are of a trivial nature, in some sense).

2. A piece of conceptual analysis: faster than light motion

Our main aim is more ambitious than providing a complete axiomatization for the kinematics of special relativity. Our complete axiomatization is only a byproduct. Our aim is to provide an analysis of the logical structure of special relativity (or in other words, giving a conceptual analysis for relativity in a precise, explicit and transparent logical framework). To this end we have to start with a list of axioms and a completeness theorem but the emphasis is on what comes beyond these.

Many of the efforts reported here are strongly connected to what is called "reverse geometry" in Pambuccian [40]. At least many of the goals of the presently reported approaches can be interpreted as doing reverse geometry for the geometry of space-time.

To illustrate what we mean by the logical analysis of relativity, we choose the much debated topic of *faster than light motion*. We refer to this as the *No FTL* conjecture where "*No FTL*" abbreviates the conjecture that faster than light motion is impossible (in relativity). The issue

[8]This is for $n > 2$, for $n = 2$ we have to add two axioms as in Thm.1.4. Roughly, by results like Thm.1.2, $\mathbf{Specrel}_0$ and **Specrel** correspond to what in Malament [35], Hogarth [26] are called "*causal space-time*" and "*space-time*" respectively, while the time-oriented version $\mathbf{Specrel} \cup \{\mathsf{f}_{mk}(\langle 1, 0, \ldots, 0\rangle)_1 > 0\}$ of **Specrel** corresponds to "*time-oriented space-time*". Cf. [4].

whether faster than light (*FTL*) observers or bodies can, in principle, exist is being seriously debated even today, cf. e.g. [20] or Matolcsi-Rodriguez [36]. A large part of the literature claims that *No FTL* is an axiom (of relativity) while another substantial part wants to get rid of *No FTL* and they maintain that *No FTL* is *not* true (i.e. that *FTL is* possible).

At the same time, in the area connecting relativity with quantum mechanics, the *No FTL* conjecture plays an essential role, e.g. the so called Einstein-Podolsky-Rosen paradox is based on assuming *No FTL*. Many authors think that *No FTL* is the essence of relativity as is suggested by the subtitle "Physics according to Newton – A world with no speed limit" in Kogut's book on relativity.

This controversy in the literature[9] makes *No FTL* an ideal testing-ground for logic in relativity. E.g. we can study which potential axioms of (special) relativity imply *No FTL*, what is implied by *No FTL*, how to obtain the most "conservative" modification of the axiom system such that it will not imply *No FTL* etc. We will return to "*No FTL*" under conditions more general than special relativity in section 4.

Let "No FTL" abbreviate the formula saying that no observer k can move *faster than light* relative to any other observer m, formally it abbreviates

$\mathsf{speed}_m(k) \leq 1$ when $m, k \in \mathsf{Ob}$.

The following is a corollary of Theorem 1.2(i):

Corollary 2.1 $\mathbf{Specrel_0} \models \mathsf{No\ FTL}$, *if* $n > 2$.

Corollary 2.1 implies that if we do not want to have No FTL as a theorem in special relativity, then we have to give up or weaken at least one of the axioms in $\mathbf{Specrel_0}$. However, even those authors who debate the status of "*No FTL*" accept all axioms of $\mathbf{Specrel_0}$.

Corollary 2.1 above can considerably be improved. Namely, if we derive "No FTL" from a weaker subsystem of $\mathbf{Specrel_0}$, we get a stronger, more interesting theorem. In particular, we get information about what other usually accepted axioms we also have to give up if we want to permit "*FTL*" travel in special relativity. Indeed, $\mathbf{Specrel_0}$ can be replaced by the much weaker subsystem **Locrel** in Cor.2.1 above, see Theorem 4.4 herein. Similar results are proved in Madarász-Tőke [33, Thm.4],[32, Thm.s3-5],[34],[31, Thm.3.2.13, p.118.], [5, Thm.4.3.24, p.497].

[9] The above-mentioned controversy is documented in [31, §2.7]. Cf. also Lewis [30, pp.67-80] and Novikov [39].

In [31, pp.89-90] it is also shown that No FTL does not follow from Einstein's *Special Principle of Relativity* (SPR) going back to Galileo, in 2-dimensional space-time, contrary to what Einstein claims in [12, pp.126-127]. In [5, §3.4.2, Thm.3.4.22, Thm.4.3.25] it is also studied how to refine the axioms of special relativity in order to make No FTL independent from the rest of the axioms. I.e. variants of special relativity are elaborated there in which *FTL is* possible (i.e. where No FTL is no more a theorem). In this way the approaches reported here intend to also contribute to the kind of research that wants to experiment with admitting *FTL* motion in relativity. For a thorough discussion of the status and literature of FTL cf. e.g. [31, §2.7, pp.70-73 (especially footnote 163)], and Lewis [30, pp.67-80, 212-213].

Ax [7] adds No FTL to his list of axioms for special relativity and he raises the question whether his axiom system is redundant. Here we will give an answer to Ax's question which at the same time will refute a claim of Mundy's for space-times of more than 2 dimensions. Namely, we will prove that No FTL is a logical consequence of Ax's remaining axioms (hence Ax's system *is* redundant), and will prove that Mundy's axioms in [38] do entail No FTL when $n > 2$. Intuitively, we conclude that No FTL is not an axiom but a theorem of (special) relativity.[10]

Now we turn to applying Cor.2.1 to answering questions in Ax [7], Mundy [38]. Ax [7] contains a finite first-order axiomatization Σ of special relativity. In this axiom system, one of the axioms, namely **AxC4**, states explicitly that all observers move slower than light. We recall **AxC4** in section 5.

$\Sigma - \{\mathbf{AxC4}\}$ denotes the set of axioms that remain of Σ after **AxC4** is deleted from it.

THEOREM 2.1 *In the axiom-system Σ in Ax* [7]*, the axiom* **AxC4** *stating that all observers move slower than light is superfluous, i.e.*

$$\Sigma - \{\mathbf{AxC4}\} \models \mathbf{AxC4}.$$

Proof-outline: Let $\mathsf{U} = \langle U, \mathsf{P}, \mathsf{S}, \mathsf{T}, \mathsf{R}\rangle$ be a model of $\Sigma - \{\mathbf{AxC4}\}$. Here, P denotes the set of "particles" and S denotes the set of "light-signals". These correspond, roughly, to our Ob and Ph, cf. section 5. Using Tarski's first-order axiomatization of Euclidean geometry, on pp. 531-532 Ax constructs a Euclidean ordered field F and to any $a \in \mathsf{P}$ a bijection $\sigma_a : {}^4\mathsf{F} \longrightarrow \mathsf{Events}$, where Events corresponds roughly to our

[10] A similar statement can be proved about general relativity, too, but there more work is needed for formalizing the *No FTL* conjecture in a precise, purely logical form; cf. sections 3,4 and Thm.4.4.

set of events. In this construction, Ax uses **AxC4** three times. With some ingenuity, one can replace the first and last uses of **AxC4** by the uses of the axioms **T1** and **AxC1** of Σ respectively. The second use of **AxC4** is not needed for the construction itself, only to ensure a specific property. Now, using the above σ_a, one can construct a model $\mathsf{M} = \langle \mathsf{B}, \mathsf{Ob}, \mathsf{Ph}, \mathsf{F}, \mathsf{W} \rangle$ of our language and check that $\mathsf{M} \models$ **Specrel**$_0$. In Ax's axiomatization, $n = 4$. By Corollary 2.1 then $\mathsf{M} \models$ No FTL. Using the definition of M, this means that $\mathsf{U} \models$ **AxC4**. QED

On the other hand, the axiomatization in Mundy[11] [38] does not contain an axiom explicitly stating "*No FTL*". But then, Mundy claims that there are models of his axiom system $\mathcal{T}$ in which there exist FTL observers. On p.43 he writes: "Therefore the only line-type information left open by the theories $\mathcal{T}$ and $\mathcal{T}'$ is which if any of the lines on or outside of the light cone are of type T, i.e. are possible paths of inertial motion. In physical terms this amounts to asking whether inertial motion can proceed at a speed equal to or greater than that of light. My contention is that nothing in either the classical or the special relativistic space-time theories provides any answer to this question. The evidence for this is that the theories $\mathcal{T}$ and $\mathcal{T}'$ seem to formalise adequately the physical content of those space-time theories, and yet do not fix an answer to this question."

THEOREM 2.2 *Let $\mathcal{T}$ be the axiom system in Mundy* [38]*. In any model of $\mathcal{T}$ all the lines in T (i.e. the so-called time-like lines) are within the light-cones. When $c \neq 0$, they are strictly within the light-cones. Hence $\mathcal{T} \models$ "No FTL".*

Proof-outline: The proof-idea is very similar to the previous one. Let M be a model of $\mathcal{T}$. (Assume that $c \neq 0$. Mundy does not seem to be aware of the fact that $\mathcal{T}$ allows $c = 0$. But one can show that if $c = 0$ in a model of $\mathcal{T}$, then the time-like and light-like lines coincide, hence we are done.) In [38, pp. 40-42], Mundy constructs co-ordinate systems for each time-like line in T, based on which one can construct a model M' of our language. One can check then that $\mathsf{M}' \models$ **Specrel**$_0$, and so $\mathsf{M}' \models$ No FTL by our Corollary 2.1 (since $n = 4$ in [38]). This means that there are no time-like lines outside the light-cones. That there are no time-like lines on the light-cone follows from [3, Prop.1]. QED

[11]Mundy's work "comes" from the seminars of Suppes on the topic of logic and relativity. Suppes (1974) credits his motivation for this line of research to interactions with Tarski.

We hope the above example (of analyzing *No FTL*) illustrates how analyzing the logical structure works in the *theory of inertial observers* (special relativity proper).

A relatively comprehensive analysis of the theories **Specrel**, $\textbf{Specrel}_0$ and their fragments can be found in [5], [31]. An investigation of the so-called "why-type" questions is also found there. Below we turn to going beyond special relativity. We will make two steps in this direction: (i) admitting accelerated observers (section 3), and (ii) making the theory local (section 4).

3. Axioms for accelerated observers, in FOL

In **Specrel** we restricted attention to inertial observers. It is a natural idea to generalize the theory to including accelerated observers as well. We will refer to such a generalized theory as a theory of accelerated observers. It is explained in the classic textbook [37, pp.163-165] that the study of accelerated observers can be regarded as a natural first step (from special relativity) towards general relativity.

The most important axiom for accelerated observers will state that at each moment of his life-time, the accelerated observer "sees" the world near him and for a short while like some inertial observer does, co-moving with him. I.e.: at every point p on the life-line of an accelerated observer k, there is an inertial observer m such that in a small enough neighborhood of p the observers k and m agree, i.e. they "see" the world roughly the same way there. We now begin to formalize this.

The *sphere* with center p and radius $\varepsilon \in \mathsf{F}$ is $S(p,\varepsilon) \stackrel{d}{=} \{q \in {}^n\mathsf{F} : |q-p| \leq \varepsilon\}$ and F^+ denotes the set of strictly positive members of F.

We say that m, k are *co-moving observers at* $q \in {}^n\mathsf{F}$, in symbols $\mathsf{comove}(m,k,q)$, if (i)-(iii) below hold for m, k as well as for k, m.

(i) $q \in \mathsf{tr}_m(k) \cap \mathsf{tr}_m(m)$,

(ii) f_{mk} is an injective function on $S(q,\eta)$ for some $\eta \in \mathsf{F}^+$, and

(iii) $\forall \varepsilon \in \mathsf{F}^+ \quad \exists \delta \in \mathsf{F}^+ \quad \forall p \in S(q,\delta) \quad |p - \mathsf{f}_{mk}(p)| \leq \varepsilon\, |p-q|$.

This notion of co-moving observers matches our intuition when $\mathsf{tr}_m(m)$ agrees with $\mathsf{tr}_k(k)$ in a neighborhood of q. $\textbf{AxSelf}^-$ below will ensure this.

To speak about accelerated observers, we introduce a new unary relation symbol Ib into our language. We will read "$\mathsf{Ib}(x)$" as "x is an *inertial body*", and then "accelerated observer" will mean "not (necessarily) inertial observer".

We are ready now for stating our main axiom for accelerated observers.

AxAcc At any point on the life-line of any observer k there is a co-moving inertial observer, i.e.

$$(\forall k \in \mathsf{Ob})(\forall q \in \mathsf{tr}_k(k))(\exists m \in \mathsf{Ob} \cap \mathsf{Ib})\mathsf{comove}(k, m, q) \ .$$

In models of **Specrel**, the world-view transformations f_{mk} are bijections everywhere defined on ${}^n\mathsf{F}$. Experimenting with constructing world-views for accelerated observers suggests that it is not natural to assume that the domain of f_{mk} is the whole ${}^n\mathsf{F}$ when k is truly accelerated and m is inertial, cf. e.g. [37, §6.6, p.173 (Fig.6.4)].[12] Therefore we relax the assumption that the accelerated observer k sees some body in each point of ${}^n\mathsf{F}$, i.e. we relax the condition that the *co-ordinate domain* of observer m is the whole ${}^n\mathsf{F}$, for every observer m. (This assumption is implicit in **Specrel**, cf. e.g. **AxPh**.) The next section is devoted to this aspect of generalizing our theory towards general relativity (where f_{mk} is only a partial function on ${}^n\mathsf{F}$).

AxSelf$^-$ A (perhaps accelerated) observer sees himself in his own co-ordinate system on that part of the time-axis where he sees a body at all, and this part is nonempty,

$$\mathsf{tr}_m(m) = \bar{t} \cap \{p \in {}^n\mathsf{F} : \exists b \mathsf{W}(m, b, p)\} \neq \emptyset.$$

In sections 1,2 we had only inertial observers (see e.g. **AxLine**), therefore, for brevity, we said "observer" in place of "inertial observer". For inertial observers we continue to use the axiom system **Specrel**, but since we now have accelerated observers as well, we will have to restrict **Specrel** to inertial observers. If **Ax** is any axiom, **Ax$^{\mathbf{in}}$** will denote the axiom we get by restricting it to inertial observers. E.g.

AxSelf$^{\mathbf{in}}$ $(\forall m, p)\Big((\mathsf{Ib}(m) \wedge \mathsf{Ob}(m) \wedge \mathsf{F}(p_1)) \rightarrow [\mathsf{W}(m, m, p) \leftrightarrow (p_2 = 0 \wedge \ldots \wedge p_n = 0)]\Big).$

We define

$$\mathbf{Specrel^{in}} \stackrel{d}{=} \{\mathbf{AxSelf^{in}}, \mathbf{AxPh^{in}}, \mathbf{AxEvent^{in}}, \mathbf{AxSym^{in}}, \mathbf{AxLine^{in}}, \mathbf{AxOb^{in}}, \mathsf{AxField}\}.$$

[12] Actually, if we assume that "radar distance" between vertical coordinate-lines is constant and we use 1/2-radar simultaneity for setting up the co-ordinate system for each accelerated observer k, then we cannot have the whole ${}^n\mathsf{F}$ as the co-ordinate domain of all accelerated observers k, assuming enough of them exist.

To be able to use **AxAcc** in models where the field-reduct is different from the real line $\Re$, we will need a kind of "induction" axiom schema. It will state that every parametrically definable subset of the field-reduct has a supremum if bounded and nonempty; as follows.

Let φ be a formula in the first-order language of our models $\mathsf{M} = \langle \mathsf{B}, \mathsf{Ob}, \mathsf{Ph}, \underline{\mathsf{F}}, \mathsf{W}\rangle$, and let $t, a_1, \ldots, a_m$ be all the free variables of φ.

Sup_φ The subset $\{t \in \mathsf{F} : \varphi(t, a_1, \ldots, a_m)\}$ of the field-reduct defined by φ when using $a_1, \ldots, a_m$ as fixed parameters has a supremum if it is bounded and nonempty, i.e.

$$(\forall a_1, \ldots a_m)[\exists t\varphi \wedge (\exists b \in \mathsf{F})(\forall t \in F)(\varphi \rightarrow t \leq b) \;\rightarrow\; (\exists d \in \mathsf{F})(\forall b \in \mathsf{F})\left((\forall t \in \mathsf{F})(\varphi \rightarrow t \leq b) \leftrightarrow d \leq b\right)].$$

Note that having $a_i \in \mathsf{F}$ and $a_j \in \mathsf{B}$ in the *same* formula φ is permitted here.

$\mathsf{IND} \stackrel{d}{=} \{\mathsf{Sup}_\varphi : \varphi$ is a first-order formula in our language$\}$.

$\mathbf{Accrel} \stackrel{d}{=} \mathbf{Specrel^{in}} \cup \{\mathbf{AxAcc}, \mathbf{AxSelf^{-}}\} \cup \mathsf{IND}$.

We note that if the field-reduct of a model M is the real line $\Re$, then $\mathsf{M} \models \mathsf{IND}$.

Accrel is at the heart of the theory of accelerated observers, in some sense, cf. e.g. [5, §8] together with [37, §6]. Some interesting statements of relativity can already be derived from it at the present point. As an example we show that the so-called *Twin Paradox* can be naturally formulated and proved in **Accrel**. See Figure 1. More importantly, the details of the Twin Paradox (e.g. who sees what, when) can be analyzed with the clarity of logic, cf. [5, pp.139-150] for part of such an analysis.

Intuitively, the *Twin Paradox* says that if one of two twin brothers leaves the other (accelerating) and returns to him later, the brother who stayed behind will be *older* at the time of their reunion. That is, more time has passed for the "non-accelerating" brother than for the traveling one.

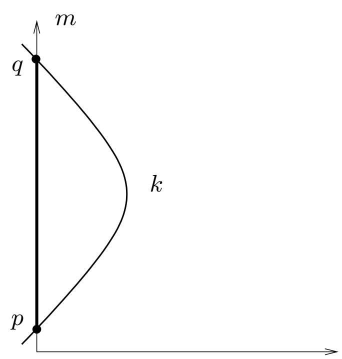
Twinp $(\forall m, k \in \mathsf{Ob})(\forall p, q, p', q' \in \bar{t})[(\mathsf{Ib}(m) \wedge p_1 < q_1 \wedge p'_1 < q'_1 \wedge \mathsf{meet}(m, k, p, p', q, q') \wedge \mathsf{depart}(m, k, p, q) \wedge \mathsf{live}(k, p', q')) \rightarrow q_1 - p_1 > q'_1 - p'_1]$, where

$\mathsf{meet}(m, k, p, p', q, q')$ denotes $\mathsf{ev}_m(p) = \mathsf{ev}_k(p') \wedge \mathsf{ev}_m(q) = \mathsf{ev}_k(q')$,

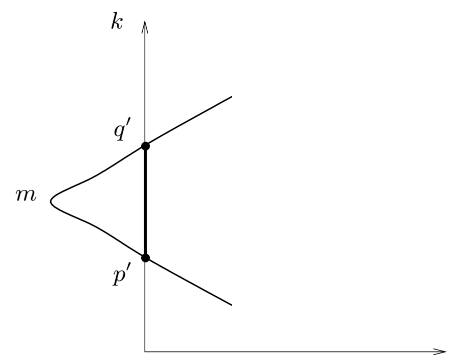
$\mathsf{depart}(m, k, p, q)$ denotes $(\exists r \in \bar{t})(p_1 < r_1 < q_1 \wedge r \notin \mathsf{tr}_m(k))$, and

$\mathsf{live}(k, p', q')$ denotes $\{r \in \bar{t} : p'_1 < r_1 < q'_1\} \subseteq \mathsf{tr}_k(k)$.

The following is from Székely [51].



m: "non-moving" (inertial) brother



k: traveling (accelerated) brother

Figure 1. The "twin paradox".

THEOREM 3.1

(i) **Accrel** $\models$ **Twinp**.

(ii) For any Euclidean ordered field F *different from the real line there is* $\mathsf{M} \models$ (**Accrel** $-$ IND) *with field-reduct* F *such that* $\mathsf{M} \not\models$ **Twinp**.

Similar results and investigation apply to the effect of gravity on clocks (also known as the *Tower Paradox*). For the time being, more concrete formulations, details etc. are available in [5] using some axioms extra to our present **Accrel**. We conjecture that the axioms extra to **Accrel** in [5] can be either eliminated or significantly simplified. (Concerning the above effect of gravity on clocks, we note that this effect is at the heart of the "science-fiction"-like behaviour of certain black holes.)

Adding the axiom schema IND to our axioms in **Accrel** represents a first step in the direction pursued in the so-called nonstandard-time logics of time represented by [43], [2] and based on Feferman [14]. Feferman's paper is devoted to reformulating everyday mathematical practice in the framework of many-sorted first-order logic.[13] All this enables us to "import" just as much of e.g. mathematical analysis into our first-order theory **Accrel** of accelerated observers as we need. Explicit and

[13]Other works in this direction carry the adjective "weak-second-order" or "Henkin-style" to indicate that in reality they use only many-sorted first-order logic. An example is weak second-order analysis. In [Takeuti, G. Proof theory, North-Holland, 1987] (Chap.3) the adjective "weak or Henkin-style" is dropped from "weak-second-order" theories.

detailed elaborations of these ideas to situations similar to our present one (theory of accelerated observers) can be found in the above quoted [43],[2] and in the works quoted therein. In developing **Accrel** further, and in related work, we will adopt the methods of the works just quoted to the framework of **Accrel** (extended with a preudo-second-order sort).

The logical analysis of the relativity theory **Accrel** is an intensively pursued research topic, cf. e.g. [5], [51]. For lack of space, here we cannot present this subject to the extent it deserves.

4. Localizing relativity

We mentioned in the previous section that it is not natural to assume that the co-ordinate domain of a truly accelerated (i.e. not inertial) observer would always be the whole ${}^n\mathsf{F}$. Motivated by this, and by other considerations that come up in developing general relativity, it is useful to investigate generalizations of **Specrel** and **Accrel** in which the co-ordinate domains of the observers are only subsets of ${}^n\mathsf{F}$. These theories are called local versions of **Specrel** and **Accrel**, respectively (where "local" is used in the same sense as in general relativity). Here we illustrate these investigations by a few examples taken from Madarász-Tőke [34],[32] and from [5, §4.9]. Localization as a step towards general relativity theory is used also in Latzer [28] and Busemann [10].

To describe the new, localized theories, first we describe what their typical models are like. Let $\mathsf{M} = \langle \mathsf{B}, \mathsf{Ob}, \mathsf{Ph}, \underline{\mathsf{F}}, \mathsf{W}\rangle$ be a model and let O be a set of subsets of ${}^n\mathsf{F}$; we define the model M^- "localized" or "relativized" to O as follows. Intuitively, for each $D \in O$ and $m \in \mathsf{Ob}$ we will have a new observer $\langle m, D\rangle$ such that the world-view of $\langle m, D\rangle$ is that of m restricted to the "co-ordinate domain" D. Formally, let $\mathsf{M}^- = \langle \mathsf{B}^-, \mathsf{Ob}^-, \mathsf{Ph}^-, \underline{\mathsf{F}}^-, \mathsf{W}^-\rangle$ where

$$\mathsf{B}^- \stackrel{d}{=} \mathsf{B} \times O, \quad \mathsf{Ob}^- \stackrel{d}{=} \mathsf{Ob} \times O, \quad \mathsf{Ph}^- \stackrel{d}{=} \mathsf{Ph} \times O, \quad \underline{\mathsf{F}}^- \stackrel{d}{=} \underline{\mathsf{F}},$$

$$\mathsf{W}^-(\langle m, D\rangle, \langle b, D'\rangle, p) \stackrel{d}{\Leftrightarrow} (p \in D \land \mathsf{W}(m, b, p)).$$

We now begin to give axioms suitable for these localized models. Throughout this section , $n \geq 2$ is arbitrary. We will always assume $\mathsf{AxField}$, and we will not indicate $\mathsf{AxField}$ explicitly in the theorems.

For simplicity, temporarily, we will talk about "inertial" observers only, so we will not use the new relation symbol Ib and "in our mind" all observers will be inertial as in the axiom system **Specrel**. Later, of course, we will extend our localization procedure from **Specrel** to **Accrel**, since it is the localized version **LocAccrel** of **Accrel** which brings us closer to general relativity.

If **Ax** is an axiom, then $\mathbf{Ax}^-$ will denote the axiom we get from **Ax** by "localizing", or "relativizing" it, and then usually $\mathbf{Ax}^{--}$ will denote an even weaker axiom. In particular, $\mathbf{Ax}^-$ will be such that if $\mathsf{M} \models \mathbf{Ax}$ then $\mathsf{M}^- \models \mathbf{Ax}^-$. (We will always assume that the elements of O are open and that they intersect the time-axis. We call a set $D \subseteq {}^n\mathsf{F}$ *open* if $(\forall p \in D)(\exists \varepsilon \in \mathsf{F}^+)S(p,\varepsilon) \subseteq D$.)

When f is a function, $\mathsf{Dom}(f)$ denotes its domain. The *co-ordinate domain* $\mathsf{CD}(m)$ of an observer m is defined as $\mathsf{Dom}(\mathsf{f}_{mm})$, i.e.

$$\mathsf{CD}(m) \stackrel{d}{=} \{p \in {}^n\mathsf{F} \;:\; (\exists b \in \mathsf{B})\mathsf{W}(m,b,p)\}\,.$$

$\mathbf{AxLine}^{--}$ The traces of observers and photons are subsets of straight lines, but they must be restrictions of lines to the co-ordinate domain (or empty),

$$(\forall m \in \mathsf{Ob})(\forall h \in \mathsf{Ob} \cup \mathsf{Ph})(\exists \ell \in \mathsf{Lines})(\mathsf{tr}_m(h) = \ell \cap \mathsf{CD}(m) \text{ or } \mathsf{tr}_m(h) = \emptyset).$$

$\mathbf{AxOb}^{--}$ Each point in the co-ordinate domain has a neighborhood and a "speed threshold" λ such that within this neighborhood each line slower than λ is the life-line of an observer,

$$(\forall m \in \mathsf{Ob})(\forall p \in \mathsf{CD}(m))(\exists \varepsilon, \lambda \in \mathsf{F}^+)(\forall \ell \in \mathsf{Lines}) \Big((\mathsf{slope}(\ell) < \lambda \;\wedge\; \ell \cap \mathsf{S}(p,\varepsilon) \neq \emptyset) \rightarrow\; (\exists k \in \mathsf{Ob})\emptyset \neq \mathsf{tr}_m(k) \subseteq \ell\Big).$$

AxOpen $(\forall m, k \in \mathsf{Ob})(\mathsf{Dom}(\mathsf{f}_{mk})$ is an *open* set).

Notice that the "speed threshold" λ in $\mathbf{AxOb}^{--}$ can vary from point to point.

If F is a field then coll denotes the ternary *collinearity* relation on ${}^n\mathsf{F}$, i.e. $\mathsf{coll}(p,q,r)$ iff $(\exists \ell \in \mathsf{Lines})\{p,q,r\} \subseteq \ell$. The *affine structure* $\mathcal{A} = \langle {}^n\mathsf{F}, \mathsf{coll}\rangle$ can be extended (uniquely) to an n-dimensional *projective structure* $\mathcal{P} = \langle \overline{{}^n\mathsf{F}}, \mathsf{coll}\rangle$ the natural way (${}^n\mathsf{F} \subseteq \overline{{}^n\mathsf{F}}$ etc.), see any textbook on projective geometry (or e.g. [22]). Throughout, by a *$\mathcal{P}$-collineation* we understand an automorphism of the projective structure $\mathcal{P}$ and by an *$\mathcal{A}$-collineation* we understand an automorphism of the affine structure $\mathcal{A}$. In other words, an $\mathcal{A}$-collineation is a permutation of ${}^n\mathsf{F}$ which preserves lines. It is known that these are exactly the bijective affine transformations composed with field-automorphism-induced mappings.

The next theorem (which is Thm.1 in [32]) says that, locally, the world-view transformations are $\mathcal{P}$-collineations in models of $\mathbf{AxLine}^{--}$, $\mathbf{AxOb}^{--}$, **AxOpen**. This is a rather strong Alexandrov-Zeeman type theorem.

THEOREM 4.1 *Assume* **AxLine**$^{--}$, **AxOb**$^{--}$, **AxOpen**. *Let* $m, k \in \mathsf{Ob}$ *and* $p \in \mathsf{Dom}(\mathsf{f}_{mk})$. *Then there is a unique* $\mathcal{P}$*-collineation* g *such that* f_{mk} *agrees with* g *on* $S(p, \varepsilon)$ *for some* $\varepsilon \in \mathsf{F}^+$. *In particular, the* f_{mk}*'s preserve* coll *and* $\neg\mathsf{coll}$ *locally.*

In the above theorem, $\mathcal{P}$-collineation cannot be replaced by $\mathcal{A}$-collineation. We get $\mathcal{A}$-collineation, however, if we add axioms about photons. Notice that so far, nothing has been used about photons. We will assume that the photon-traces that cross a given point form a cone, called *light-cone*. We are going to formalize this.

The set of *spatial directions* dir is defined as

$$\mathsf{dir} :\stackrel{\mathsf{d}}{=} \{\mathsf{d} \in {}^{\mathsf{n}-1}\mathsf{F} \ : \ \mathsf{d} \neq \overline{0}\}.$$

Assume $m \in \mathsf{Ob}$, $b \in \mathsf{B}$, $d \in \mathsf{dir}$. We say that b *moves in direction* d *as seen by* m iff $(\forall p, q \in \mathsf{tr}_m(b))(\exists \lambda \in \mathsf{F})$
$[\langle q_2, \dots, q_n\rangle - \langle p_2, \dots, p_n\rangle = \lambda d \ \wedge \ (q_1 > p_1 \ \rightarrow \ \lambda \geq 0)]$.
We note that $\mathsf{speed}_m(b)$ is meaningful also when $\mathsf{tr}_m(b)$ is an at least two-element subset of a line. In the following we will use $\mathsf{speed}_m(\mathsf{ph})$ in this sense. Recall that $\mathsf{speed}_m(\mathsf{ph}) = \infty$ abbreviates that $\mathsf{tr}_m(\mathsf{ph})$ is "horisontal" (as introduced at the beginning of section 1).

Ax∃Ph From any point $p \in \mathsf{CD}(m)$ in any direction there is a photon moving forwards in that direction,

$$(\forall m \in \mathsf{Ob})(\forall p \in \mathsf{CD}(m))(\forall d \in \mathsf{dir})(\exists \mathsf{ph} \in \mathsf{Ph})$$
$$[\,p \in \mathsf{tr}_m(\mathsf{ph}) \wedge \ (\mathsf{ph} \text{ moves in direction } d \text{ as seen by } m)].$$

AxItr The speed of light is direction-independent locally,

$$(\forall \mathsf{ph}, \mathsf{ph}' \in \mathsf{Ph})\,(\mathsf{tr}_m(\mathsf{ph}) \cap \mathsf{tr}_m(\mathsf{ph}') \neq \emptyset \rightarrow \mathsf{speed}_m(\mathsf{ph}) = \mathsf{speed}_m(\mathsf{ph}')).$$

In effect, the photon-traces that cross a given $p \in \mathsf{CD}(m)$ form a cone, called *light-cone*. Notice that the speed of light—the angle (or "openness" or "width") of the light-cone—may differ from point to point. Here, *Itr* abbreviates "isotropy".

AxFin The speed of each photon is nonzero and finite,

$$(\forall m \in \mathsf{Ob})(\forall \mathsf{ph} \in \mathsf{Ph})\,(0 < \mathsf{speed}_m(\mathsf{ph}) \neq \infty \text{ or } \mathsf{tr}_m(\mathsf{ph}) = \emptyset).$$

THEOREM 4.2 *Assume* **AxLine**$^{--}$, **AxOb**$^{--}$, **AxOpen**, **Ax∃Ph**, **AxItr**, **AxFin**. *Assume* $m, k \in \mathsf{Ob}$ *and* $p \in \mathsf{Dom}(\mathsf{f}_{mk})$. *Then there is a unique* $\mathcal{A}$*-collineation* g *such that* f_{mk} *agrees with* g *on* $S(p, \varepsilon)$, *for some* $\varepsilon \in \mathsf{F}^+$. *Hence the* f_{mk}*'s preserve parallelism,* coll *and* $\neg\mathsf{coll}$ *locally.*

Theorem 4.3 below, taken from Lester [29, p.929], shows that the assumption **AxOb**$^{--}$ cannot be omitted from Theorem 4.2 above, even if we replace **AxItr** with the stronger **AxE** defined below. It also shows that the localized version of the Alexandrov-Zeeman theorem Thm.1.3 fails to hold.

AxE $(\forall m \in \mathsf{Ob})(\forall \mathsf{ph} \in \mathsf{Ph})\mathsf{speed}_m(\mathsf{ph}) = 1$.

THEOREM 4.3 *For any $n \geq 2$ there is a model of* $\{$**AxLine**$^{--}$, **AxOpen**, **Ax∃Ph**, **AxE**, **AxSelf**$\}$ *with $m, k \in \mathsf{Ob}$ such that $\mathsf{Dom}(\mathsf{f}_{mk}) \neq \emptyset$ and for any $p \in \mathsf{Dom}(\mathsf{f}_{mk})$ and any $\varepsilon \in \mathsf{F}^+$ the world-view transformation f_{mk} does not preserve* coll *on $\mathsf{S}(p, \varepsilon)$, i.e. there are $a, b, c \in \mathsf{S}(p, \varepsilon)$ such that $\mathsf{coll}(a, b, c)$ but $\neg\mathsf{coll}(\mathsf{f}_{mk}(a), \mathsf{f}_{mk}(b), \mathsf{f}_{mk}(c))$. I.e., the collinear points a, b, c the collinearity of which is not preserved are densely located everywhere in $\mathsf{Dom}(\mathsf{f}_{mk})$.*

It might be interesting to notice that while studying **Specrel** and its fragments (cf. section 1) assuming or omitting axiom **AxOb** for $n > 2$ did not matter too much, in the present, local theory (or localized version of **Specrel**) **AxOb**$^-$ does matter by Theorems 4.1–4.3.

We now turn to proving a "no FTL" theorem in the local setting. We will replace **AxItr** with a much weaker assumption, **AxP1**, but at the same time we will weaken-and-strengthen **AxOb**$^{--}$ to requiring that the observer-traces "fill" the light-cones.

AxP1 The speed of light is unique and well-defined in each direction at each location,

$(\forall \mathsf{ph}, \mathsf{ph}' \in \mathsf{Ph})[\mathsf{ph}$ and ph' move in the same direction as seen by $m \rightarrow$ $(\mathsf{tr}_m(\mathsf{ph}) = \mathsf{tr}_m(\mathsf{ph}')$ or $\mathsf{tr}_m(\mathsf{ph}) \cap \mathsf{tr}_m(\mathsf{ph}') = \emptyset)]$.

Basically, this is the first-order logic formalization of Friedman's principle (P1) in [19, p.159].

AxPh^{--} $\stackrel{d}{=}$ **Ax∃Ph** $\wedge$ **AxP1** $\wedge$ **AxFin**.

AxOb$^-$ There are observers on lines which are slower than light locally,

$(\forall m \in \mathsf{Ob})(\forall \mathsf{ph} \in \mathsf{Ph}, p \in \mathsf{tr}_m(\mathsf{ph}))(\forall 0 \leq \lambda < \mathsf{speed}_m(\mathsf{ph}))(\exists k \in \mathsf{Ob})$ $[p \in \mathsf{tr}_m(k)$, $\mathsf{speed}_m(k) = \lambda$, and ph, k move in the same direction as seen by $m]$.

We will also use a weakened version of the local version of **AxEvent** formalized as follows. Assume $k, h \in \mathsf{Ob}$. Then we say that k is a *brother* of h iff $(\forall m \in \mathsf{Ob})$ $\mathsf{tr}_m(k) = \mathsf{tr}_m(h)$. In a model M^- relativized to a set

O of subsets of ${}^n\mathsf{F}$, if $k \in \mathsf{Ob}$, then the new observers $\langle k, D\rangle, \langle k, D'\rangle$ are brothers. They may "see" different events.

AxEvent^{--} If m sees an event happening to k, some brother of k sees it, too,

$(\forall m, k \in \mathsf{Ob}, p \in \mathsf{tr}_m(k))(\exists h \in \mathsf{Ob}, q \in {}^n\mathsf{F})[h$ is a brother of k and $\mathsf{ev}_m(p) = \mathsf{ev}_h(q)]$.

$$\textbf{Locrel} \stackrel{d}{=} \{\textbf{AxSelf}^-, \textbf{AxLine}^{--}, \textbf{AxPh}^{--}, \textbf{AxOb}^-, \textbf{AxEvent}^{--}, \textbf{AxOpen}, \mathsf{AxField}\}.$$

Now, **Locrel** is a localized version of **Specrel**$_0$. FTL abbreviates faster than light. Let $k, m \in \mathsf{Ob}$. We call k FTL w.r.t. m iff there is a $\mathsf{ph} \in \mathsf{Ph}$ such that k and ph move in the same direction as seen by m, they meet, i.e. $\mathsf{tr}_m(k) \cap \mathsf{tr}_m(\mathsf{ph}) \neq \emptyset$, and $\mathsf{speed}_m(k) > \mathsf{speed}_m(\mathsf{ph})$. **NoFTL** abbreviates the formula saying that no observer k can move *faster than light*, in the present *local* sense, relative to any other observer, i.e. it abbreviates the formula $\neg(\exists m, k \in \mathsf{Ob})[k \ \mathsf{FTL}$ w.r.t. $m]$.

THEOREM 4.4 *Assume* $n > 2$. *Then* **Locrel** $\models$ **NoFTL**.

For $n = 2$, or if we omit any assumption from **Locrel**, FTL observers do become possible.

The local version **LocAccrel** of the theory **Accrel** of accelerated observers can be obtained by adding **AxAcc**, **AxSelf**$^-$, IND to **Locrel**$^{\mathbf{in}}$ (where the latter is defined analogously to **Specrel**$^{\mathbf{in}}$).

5. Samples from the literature

Einstein named his theories of relativity after the *principle of relativity* (PR) a special case of which is SPR mentioned in section 2. Einstein derives the theory of special relativity as a consequence of SPR. (PR is analogously related to general relativity.) The principle of relativity has been playing a central role of European thinking for almost 2400 years by now, very roughly as follows. Primitive versions of PR were proposed by Heraclides of Ponticus (388-315 BC) and Aristarchus (310-230 BC). PR was "abolished" by Ptolemy (c. 140 AD), and restored by Nicole d'Oresme (c. 1350), Copernicus (c. 1510) and Galileo (c. 1600). It came again in doubt during the 1800's because the theory of electromagnetism seemed to contradict PR. Finally, Einstein restored PR (and even generalized it).[14] Physicist Max Born said about Einstein's theory:

[14] Some ideas about the principle of relativity remain open for future work to clarify, in the form of Mach's principle, cf. Barbour [8] and Gödel [21]. The first "approximately modern"

"The theory appeared to me, and it still does, the greatest feat of human thinking ...". This brief history indicates that it is worthwhile to apply the machinery of modern logic for obtaining a refined analysis of the logical structure of relativity theory, or theory of space-time.

Indeed, one could say that logic was started by the effort of axiomatizing geometry (by Euclid). This axiomatization was brought to its modern form by Tarski, who replaced Hilbert's second-order logic axiomatization by one based on purely first-order logic, cf. Tarski [53], Pambuccian [40]. But the geometry of space-time[15] is at the center of relativity, hence it is natural to base the logical analysis of relativity on (a continuation of) the Tarskian tradition of logicizing geometry.

There are several works in the literature devoted to the subject of axiomatizing special relativity, or at least its space-time geometry. To mention some: axiomatizations of special relativity have been studied in e.g. Robb 1914 [42], Reichenbach 1924 [41], Carathéodory 1924 [11], Alexandrov and his school starting with 1950 [1], [23], [56], Suppes and his school starting with 1959 [47],[48],[49], Basri 1966 [9], Szekeres 1968 [52], Ax 1978 [7], Friedman 1983 [19], Mundy 1986 [38], Goldblatt 1987 [22], Schutz 1997 [46]. Latzer [28], Busemann [10], Walker [57] contain experiments in the direction of extending the logic based, axiomatic approach towards general relativity. All this is only a small sample. There are more works listed in our bibliography and the bibliographies of these. In this section we briefly recall a few samples from the rich literature mentioned earlier. For lack of space, we cannot even come close to do justice to the works deserving attention. Rich sources of further references are Schelb [44], Schröter [45].

Guts [23] is a survey of work done in the field of axiomatizing relativity theory in and around Russia. Their axiomatizations are not yet logic-based, but they go beyond just giving completeness theorems. We already saw a theorem of this school, the Alexandrov-Zeeman theorem. Here we cite another theorem of Alexandrov to get a flavor of these results. For a more thorough survey we refer to Guts [23].

First we give an axiomatization and then the completeness theorem. A *space-time* in [23] is defined to be a system $\langle V, \langle P_a : a \in V\rangle, T\rangle$ which satisfies **A1** below.

formulation of PR is due probably to Oresme, cf. [8, §4.4 (.. the early ideas about relativity), pp.206-207] and §2.5 in [Mlodinow, L., Euclid's window. Touchstone, New York, 2002].

[15] The space-time of special relativity was geometrized by Minkowski, Minkowski-geometry is one of the so-called pseudo-Euclidean geometries. The geometry of general relativity is locally pseudo-Euclidean.

A1 V is a connected, simply-connected locally compact 4-dimensional Hausdorff space, $\langle P_a : a \in V\rangle$ is a family of subsets of V and T is a *transitive, commutative* group of homeomorphisms of V onto itself satisfying

$\{a\} \subset P_a$,

$P_x \subseteq P_y$ if $x \in P_y$,

$P_{t(a)} = t[P_a]$ for all $a \in V, t \in T$.

We note that this implies that V is homeomorphic to ${}^n\Re$ where $\Re$ is the real line.

The survey [23] emphasizes the "*causal-future-cone*" structure $\langle V, \langle P_x : x \in V\rangle\rangle$ and in this respect is related to Robb's approach (cf. [42], [22]) emphasizing the partial order structure $\langle V, \ll\rangle$ of space-time, where $x \ll y$ iff $y \in P_x$. Many authors call $\ll$ the *causality relation* and call P_x the causal future (cone) of x. The general relativistic approach of Busemann [10] also starts out with the structure $\langle V, \ll\rangle$ (then localizes it in a spirit similar to our section 4). The topology on V can usually be derived from $\ll$.

We are now going to state further axioms. If $X \subseteq V$ then ∂X and $intX$ denote the *boundary* and *interior* of X respectively and if $a \in V$ then G_a denotes the set of all permutations g of V for which $g(a) = a$ and $g[P_x] = P_{g(x)}$ for all $x \in V$.

A2 $P_x \cap \{z \in V : y \in P_z\}$ is bounded if $y \in P_x$,

A3 $(\forall x, y \in \partial P_a - \{a\})(\exists g \in G_a)g(x) = y$,

A4 $(\forall x, y \in intP_a)(\exists g \in G_a)g(x) = y$,

A5 $intP_e \neq \emptyset$ for some $e \in V$.

Motivation for this definition of space-time and for some of the axioms is given in Guts [23, pp.44-46]. The philosophy is that "space-time is a form of existence of matter". $\langle V, T\rangle$ is a 4-dimensional affine space with group T of translations and "$x \in P_y$" intends to represent the so-called causality relation $x \ll y$ with the intended meaning "action (energy-momentum) is transmitted from x to y". So P_x can be interpreted as the causal future cone of event x. Further, ∂P_x is the "light-cone" starting at event x. G_x serves to represent those world-view transformations which leave event x fixed. Axiom **A2** is interpreted as saying that the velocity of transmission of energy is finite.

Theorem 5.1 (Alexandrov [1]) *Assume* $\mathbf{A1} - \mathbf{A5}$ *and let* $e \in V$*. Then* P_e *is a closed or an open elliptic cone, and* G_e *is the homogeneous Lorentz-group with dilations. I.e. there are Cartesian coordinates* $x_1 \ldots,$ x_4 *such that if* e *has co-ordinates* $\langle 0,0,0,0\rangle$ *then*

$$P_e = \{\langle x_1, \ldots, x_4\rangle \in V : x_4^2 \geq x_1^2 + x_2^2 + x_3^2,\ x_4 \geq 0\} \text{ or}$$

$$P_e = \{\langle x_1, \ldots, x_4\rangle \in V : x_4^2 > x_1^2 + x_2^2 + x_3^2,\ x_4 > 0\}, \text{ and}$$

$$G_e = \{f : f \text{ is a linear transformation preserving Minkowski-distance composed with a dilation}\}.$$

The other axiomatizations in the list we gave at the beginning of this section are in the framework of logic more or less, though not always in a very formalized manner. Robb [42] is the earliest of these, and his work is the model or starting point of many later axiomatizations. E.g. both Ax and Mundy quote his work explicitly as the inspiration of their own axiom systems. Robb had only one binary relation in his logical language, that of the causality relation $\ll$, called often also the relation "after". He had 21 axioms.

Suppes' 1959 axiom system in [47] is in second-order logic, but later in [49] (1972) he outlined how to modify this system to be a first-order one. Here we quote the axiom system from [47].

The pair $\langle M, \mathsf{Ob}\rangle$ is called a *collection of relativistic frames* in [47] if **S1, S2** below hold.

S1 M is a set and Ob is a set of bijections between M and the set ${}^4\Re$ of four-tuples of real numbers.

If $x, y \in M$ and $m \in \mathsf{Ob}$ then define $\mu_m(x,y) \stackrel{d}{=} \mu(m(x) - m(y))$ where μ denotes the Minkowski-length (introduced earlier, in section 2).

S2 $(\forall x, y \in M)[(\exists m \in \mathsf{Ob})\mu_m(x,y) > 0 \rightarrow (\forall m, k \in \mathsf{Ob})\mu_m(x,y) = \mu_k(x,y)]$.

Theorem 5.2 (Suppes [47]) *Let* $\langle M, \mathsf{Ob}\rangle$ *be a collection of relativistic frames. Then* $m^{-1} \circ k$ *is a Poincaré-transformation for all* $m, k \in \mathsf{Ob}$*.*

Reichenbach [41] emphasized the importance of *observation-oriented* axioms. The idea is that we start out from observation-oriented axioms, then we work with these axiom systems, and during this work useful *theoretical* notions arise. Based on these theoretical notions we get a more thorough understanding, and we develop another, theoretical-notion oriented axiom system. We can call the first approach "bottom up" while the second one "top down". The first kind of axiom systems are more intuitive, while the second kind more compact mathematically. We consider Suppes' system as top down, since there is no intuition given why to think in terms of exactly the Minkowski-distance of events (as opposed to e.g. the usual Euclidean distance). The Minkowski-distance of events in the "bottom up" approaches (like ours) emerges naturally as "theoretical", defined concepts, see e.g. Theorem 1.2.

The idea is that the theoretical concepts should be first-order definable in terms of the observational-oriented concepts. Definability theory of first-order logic is very important in this process. Indeed, definability theory was started by Reichenbach (for the purposes of relativity theory), and later worked out for first-order logic by Tarski. Cf. also [16], [24], and the related literature of identifiability [25]. Madarász [31, §4.3] extended the theory of definability to defining new sorts in many-sorted logic, i.e. to defining new universes. This theory could be fruitfully used in comparing the various axiomatizations present in the literature. More on this and on the equivalence of "bottom up" and "top down" approaches can be read in [31, §4].

The axiom system of Ax [7] is the first truly first-order and observation oriented axiom system. His system is very elegant and intuitively clear. His language contains two unary relation symbols P, S for "particles" and "signals", and two binary relation symbols T, R for "transmitting a signal" and "receiving a signal". Particles are "imagined" as life-lines of bodies and signals are "imagined" as finite segments of life-lines of photons. Signals of length zero are conceived as "events". Ax has three groups of axioms, altogether 23 ones. Here we quote two typical axioms from his paper. The first axiom corresponds to our "No FTL" formula, while the second axiom corresponds to our axiom **AxOb**.

AxC4 [maximality of signal speed] For all particle a and event γ there is a unique signal β which begins at γ and reaches a,

$(\forall a \in \mathsf{P}, \gamma \in \mathsf{S})[\mathsf{Ev}(\gamma) \to (\exists! \beta)[\mathsf{Beg}(\beta) = \gamma \wedge a \mathsf{R} \beta]]$, where

$\mathsf{Ev}(\gamma)$ abbreviates $(\forall a)(a\mathsf{T}\gamma \to a\mathsf{R}\gamma)$ and
$\mathsf{Beg}(\beta) = \gamma$ abbreviates $\forall a[a\mathsf{T}\beta \to (a\mathsf{T}\gamma \wedge a\mathsf{R}\gamma)]$.

AxC5 [axiom of the limiting value of speed of light] Assume that the particle a receives the signal β in the event γ_1 and the event γ_2 occurs later on a's life-line. Then there is a particle b which receives both the beginning of β and γ_2,

$(\forall a, \beta, \gamma_1, \gamma_2, \gamma_3)[(a\mathsf{R}\beta \wedge \mathsf{End}(\beta) = \gamma_1 \wedge \mathsf{Ev}(\gamma_2) \wedge a\mathsf{R}\gamma_2 \wedge \gamma_1 \ll \gamma_2 \wedge$
$\mathsf{Beg}(\beta) = \gamma_3) \to (\exists b)(b\mathsf{R}\gamma_3 \wedge b\mathsf{R}\gamma_2)]$, where

$\mathsf{End}(\beta) = \gamma_3$ abbreviates $(\forall a)[a\mathsf{R}\beta \to (a\mathsf{R}\gamma_3 \wedge a\mathsf{T}\gamma_3)]$ and

$\gamma_1 \ll \gamma_2$ abbreviates $\mathsf{Ev}(\gamma_1) \wedge \mathsf{Ev}(\gamma_2) \wedge \gamma_1 \neq \gamma_2 \wedge (\exists a, \alpha, \beta)[a\mathsf{T}\gamma_1 \wedge \mathsf{Beg}(\alpha) = \gamma_1 \wedge \mathsf{End}(\alpha) = \mathsf{Beg}(\beta) \wedge \mathsf{End}(\beta) = \gamma_2 \wedge a\mathsf{R}\gamma_2)$.

This $\ll$ is the same causality relation as in the earlier systems.

Ax [7] thoroughly presents the intuitive, physical motivation for his approach, his choice of framework, his axioms etc. which are ambitious and satisfying even on the level of abstraction of philosophy of physics (and philosophy of science). The motivation followed by the present team coincides with the motivation given by Ax (and is very close to the one in Suppes [48]).

Goldblatt [22] contains another first-order axiomatization of Minkowski space-time, i.e. of special relativity. While Ax's language is observational-oriented, and he emphasizes this, Goldblatt's system is clearly theoretical-oriented. One of his basic notions is that of orthogonality. Both Ax and Goldblatt build on Tarski's first-order axiomatization of Euclidean geometry. In fact, Goldblatt's system is a very harmonious extension/modification to Minkowski-geometry of Tarski's system for Euclidean geometry.

The structures axiomatized by Ax and Goldblatt do not contain the (relativistic) information carried by the so-called relativistic distance or Minkowski-metric. So it seems that the first, observation-oriented axiomatization in first-order logic of special relativity with all the relevant data (including relativistic distance) was obtained in the works of the present team. The efforts, sampled in the introduction and in §§ 1-4 above, of elaborating a conceptual analysis of relativity and a kind of "reverse mathematics" for relativity theory based on a suitable system of FOL-axiomatizations like **Specrel** etc. seem to be more recent. However, reverse mathematics for non-relativistic geometry is extensive, cf. e.g. Pambuccian [40] and the references therein.

Acknowledgements. We would like to thank János Makowsky, Ildikó Sain, and Jesse Alama for helpful comments on an earlier version. Heartfelt thanks go to Victor Pambuccian for helpful suggestions and pointers to the literature. This research was supported by the Hungarian National Fund for basic research grants No's T30314, T43242, T35192, and by COST project 274.

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December 2003.

Appeared in: Non-Euclidean Geometries, ed.s: A. Prékopa and E. Molnár, Mathematics and its Applications, Springer, 2006. pp.155-185.