

A finite axiomatization of locally square cylindric-relativized set algebras*

by

H. Andréka

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Abstract

We give a finite set of equations axiomatizing the class G_n of locally square cylindric-relativized set algebras of dimension n , if n is finite. For infinite n , we give an axiomatization of the equational theory of G_n . Here G_n denotes the class of all cylindric-relativized set algebras of dimension n with unit a union of Cartesian spaces.

Let n be an ordinal. We will deal with algebras of n -ary relations. A cylindric-relativized set algebra of dimension n is an algebra of n -ary relations. In more detail, an algebra $\mathfrak{A} = \langle A, +, -, c_i, d_{ij} \rangle_{i,j < n}$ is a *cylindric-relativized set algebra* of dimension n (a Crs_n) if the following (i)-(ii) hold.

- (i) $\langle A, +, - \rangle$ is a Boolean set algebra whose elements are n -ary relations. Let V denote the greatest element of this algebra. V is called the *unit* of \mathfrak{A} . Thus $+$, $-$ denote the operations of taking union and taking complement w.r.t. V .
- (ii) The additional operations are as follows. Let $i, j < n$. Then d_{ij} is a constant and c_i is a unary operation defined as follows.

$$d_{ij} \stackrel{\text{def}}{=} \{z \in V : z_i = z_j\}.$$

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Let $x \in A$. Then x is an n -ary relation, i.e. x is a set of n -sequences. Now $c_i(x)$ is the set of those n -sequences from V which agree everywhere but on i with a sequence in x ; formally

$$c_i(x) \stackrel{\text{def}}{=} \{z \in V : (\exists s \in x)(\forall k < n, k \neq i)z_k = s_k\}.$$

Thus d_{ij} and $c_i(x)$ are n -ary relations. We require that $d_{ij} \in A$ and $c_i(x) \in A$ for all $i, j < n$ and $x \in A$.

CrS_n 's are Boolean algebras with additional unary operators c_i and constants d_{ij} (Bo_n 's, cf. [14]). In some sense, these are the most natural algebras of n -ary relations.

CrS_n denotes the class of all cylindric-relativized set algebras of dimension n , and $GS_n \subseteq G_n \subseteq D_n \subseteq CrS_n$ are subclasses of CrS_n defined by making restrictions on the biggest element (unit) of the algebra. We briefly recall their definitions below.

By a *Cartesian space* we mean the set of all U -termed n -sequences for some set U , where s is a U -termed n -sequence if $s = \langle s_i \rangle_{i < n}$ and $s_i \in U$ for all $i < n$. We will consider sequences to be functions, and an ordinal to be the set of all smaller ordinals. Thus, an n -sequence $s = \langle s_i \rangle_{i < n}$ is a function with domain $n = \{i : i < n\}$. A U -termed n -sequence s is then a function $s : n \rightarrow U$. If $\tau : n \rightarrow n$ is a transformation on n , then $s \circ \tau = \langle s_{\tau(i)} \rangle_{i < n}$ is a new sequence, namely the sequence we obtain from s by "rearranging" it according to τ . $\tau : n \rightarrow n$ is called *permutational* if it is a bijection. We call τ *finite* if τ moves only finitely many elements, i.e. $\{i < n : \tau(i) \neq i\}$ is finite.

Let \mathfrak{A} be a cylindric-relativized set algebra of dimension n with unit V . Then

$\mathfrak{A} \in D_n$ iff for every $s \in V$ and every finite nonpermutational transformation τ of n , also $s \circ \tau$ is in V .

$\mathfrak{A} \in G_n$ iff V is a union of Cartesian spaces.

$\mathfrak{A} \in GS_n$ iff V is a disjoint union of Cartesian spaces.

An equivalent definition of G_n is that $s \circ \tau \in V$ for every transformation τ of n and sequence $s \in V$.

For a class K of algebras, IK denotes the class of all isomorphic copies of elements of K . Now, $ICrS_n, ID_n, IG_n, IGS_n$ are all distinct classes, equations distinguishing them are given in [19, 20].

Let n be finite. In this paper we give a finite equational axiom system for G_n . It was already known that IG_n is axiomatizable by equations, i.e. it is a *variety* (Németi [19]), moreover it is a canonical variety, i.e. it is closed under perfect extensions (Andréka-Goldblatt-Németi [3]). These will also follow from the theorem in the present paper because the axiomatization we give contains only positive equations (i.e. Boolean negation – does not occur in the extra-Boolean equations, but Boolean product \cdot may occur in them).

If n is infinite, then we do not know whether IG_n is a variety or not. We do not even know whether IG_n is axiomatizable with first-order formulas or not. In this paper we give a finite scheme of equations axiomatizing the equational theory of G_n . This finite scheme is a natural generalization of the axiom system given for the finite case. More on IG_n can be found in [3, 4, 13, 15, 19, 20, 21].

Let now $n \geq 3$ be arbitrary. To contrast the above results, we note that $ICrs_n, ID_n, IGs_n$ are all varieties, $ICrs_n$ and IGs_n are not finitely axiomatizable, and a finite axiomatization for ID_n is given in [6]. There is a difference between non-finite (schema) axiomatizability of Gs_n and Crs_n , however: Gs_n cannot be axiomatized with any axiom system that contains only finitely many variables ([1]), while an (infinite) equational axiomatization of Crs_n which uses only two variables is given in [16] (this system is due to D. Resek and R. Thompson). Axiomatizations for Gs_n are surveyed in [9], a recent new axiom system for Gs_n is in [11]. A rich material on these classes can be found, besides the above cited references, e.g. in [7, 5, 10, 9, 17, 18].

We now turn to giving the equational axiomatization of IG_n .

Let n be a finite number and let $i, j < n$. Let Ax_{ij} be the following equation:

$$x \leq c_i c_j (s_j^i c_j x \cdot s_i^j c_i x \cdot \prod_{k < n, k \neq i, j} s_i^k s_j^i s_k^j c_k x).$$

In the above, \prod is the grouped version of \cdot , and $s_j^i(x) = c_i(d_{ij} \cdot x)$ if $i \neq j$ and $s_i^i(x) = x$. We will see from the proof of the next theorem that Ax_{ij} intuitively says that for any sequence in the unit, we can also put into the unit the sequence we obtain from it by interchanging its i 'th and j 'th elements. From the equation Ax_{ij} we will only use its following consequence:

$$x \neq 0 \quad \rightarrow \quad s_j^i c_j x \cdot s_i^j c_i x \cdot \prod_{k < n, k \neq i, j} s_i^k s_j^i s_k^j c_k x \neq 0.$$

Intuitively, this ensures that if a sequence is in x , then its permuted version can be put into $s_j^i c_i x \cdot \dots$

To extend these equations for infinite n , assume now that n is any ordinal, possibly infinite. For all $i, j < n$ and all finite subsets Γ of n let $Ax_{ij\Gamma}$ denote the following equation:

$$x \leq c_i c_j (s_j^i c_j x \cdot s_i^j c_i x \cdot \prod_{k \in \Gamma, k \neq i, j} s_i^k s_j^i s_k^j c_k x).$$

Let Ax_n be the set of all Ax_{ij} , for $i, j < n$, and let $Ax'_n = \{Ax_{ij\Gamma} : i, j < n, \Gamma \subseteq n, \Gamma \text{ finite}\}$. We will simply write Ax and Ax' if there is no danger of confusion. IG_n, HG_n denote the classes of all isomorphic copies and all homomorphic images of elements of G_n , respectively. We note that a simple finite set of equations (equation schemes in the case when n is infinite) which axiomatizes ID_n is given in [6]. Therefore Theorem 1 below gives axiom systems for IG_n and HG_n ; and these axiom systems are finite if n is finite.

Theorem 1 (Axiomatizations of IG_n and HG_n)

- (i) $IG_n = \{\mathfrak{A} \in ID_n : \mathfrak{A} \models Ax\}$, if n is finite, $n \geq 3$.
- (ii) $HG_n = \{\mathfrak{A} \in ID_n : \mathfrak{A} \models Ax'\}$ for all $n \geq 3$.
- (iii) $IG_2 = \{\mathfrak{A} \in ID_n : \mathfrak{A} \models x - d_{01} \leq c_0 c_1 (-d_{01} \cdot s_1^0 c_1 x \cdot s_0^1 c_0 x)\}$.

Proof. Let $n \geq 2$. First we show that $G_n \models Ax'$. Let $\mathfrak{A} \in G_n$ with unit V , $x \in A$ and let $s \in x$. Let $i, j, k < n$, $k \neq i, j$. Let $z = s \circ [i, j]$, where $[i, j] : n \rightarrow n$ is the permutation of the set $n = \{\ell : \ell < n\}$ which interchanges i and j and leaves all the other elements fixed. Then $z \in V$ and it is easy to check that $s \in c_i c_j (\{z\})$, $z \in s_j^i c_j x$, $z \in s_i^j c_i x$, $z \in s_i^k s_j^i s_k^j c_k x$. This shows that $\mathfrak{A} \models Ax_{ij\Gamma}$ for all $i, j < n$ and for all finite $\Gamma \subseteq n$. This shows $G_n \models Ax'$. The same argument shows that G_2 satisfies the indicated equation (notice that if $s \in x - d_{01}$, then $s_0 \neq s_1$, and so z also is in $-d_{01}$).

To show the other direction of (i)-(iii), we first assume that n is finite $n \geq 2$. The proof below is an extension of the one in [6], and it will also show some similarities with the proofs in [2].

First we make an observation. Let V be a D_n -unit. Then

(*) If $s \in V$ has a repetition, then $s \circ \tau \in V$ for all $\tau : n \rightarrow n$.

Indeed, if τ is nonpermutational, then $s \circ \tau \in V$ by the definition of a D_n -unit and since n is finite. If τ is permutational, then $s \circ \tau = s \circ \tau'$ for some nonpermutational τ' because s has a repetition. (Indeed, assume $s_i = s_j$ for $i, j < n$, $i \neq j$. Let τ' be

such that $\tau'(m) = \tau(m)$ if $m \neq \tau^{-1}(j)$, and $\tau'(\tau^{-1}(j)) = \tau(i)$.) This shows that (*) holds. By (*) we have that if a D_n -unit V is such that $s \circ \tau \in V$ for all repetition-free $s \in V$ and for all bijections τ , then V is a G_n -unit.

Let $\mathfrak{A} \in ID_n$, and assume that $\mathfrak{A} \models Ax$. We want to show that \mathfrak{A} is isomorphic to an $\mathfrak{A}'' \in G_n$.

It is proved, implicitly, in [6] that every algebra $\mathfrak{A} \in ID_n$ is isomorphic to an $\mathfrak{A}' \in D_n$ such that the unit V of \mathfrak{A}' satisfies (**) below:

(**) For all repetition-free sequences $s, z \in V$, the ranges of s and z are different, i.e. $Rng(s) \neq Rng(z)$.

Indeed, in the proof of Theorem 1 in [6], to any $\mathfrak{A} \in ID_n$ we define a set V such that \mathfrak{A} is isomorphic to $\mathfrak{A}' \in D_n$ with unit V . The construction of V is given on p.674 in [6], and conditions (b),(c),(d) on the same page ensure that (**) holds for V .

(**) means, in other words, that if $s \in V$ and s is repetition-free, then no other permuted version of s is in V . By (**), the unit of \mathfrak{A}' satisfies the weaker condition (***) below:

(***) For all repetition-free sequences $s \in V$, either $s \circ \tau \in V$ for all bijections $\tau : n \rightarrow n$ or else $s \circ \tau \notin V$ for all bijections $\tau : n \rightarrow n$ which are not the identity on n .

The idea of the following proof is that we can “throw in” the permuted versions of the repetition-free sequences in V such that “ \mathfrak{A}' will not change”. Ax_{ij} will tell us “where to put the new sequence $s \circ [i, j]$ ”. In more detail, assume that $\mathfrak{A}' \in D_n$ has unit V such that (***) holds. Let $s \in V$ be a repetition-free sequence such that “no permuted version of s is in V ”. Let $S = \{z_0, z_1, \dots, z_N\}$ be a listing of all the permuted versions of s — i.e. $S = \{s \circ \tau : \tau \text{ is a bijection of } n\}$ — such that

$$\begin{aligned} z_0 &= s \\ z_i &= z_j \circ [k, \ell] \text{ for some } j < i, k, \ell < n \\ z_i &\neq z_j \text{ if } i \neq j. \end{aligned}$$

Such a listing is possible. Then $V \cup S$ also satisfies (***) . We will “represent” \mathfrak{A}' on $V \cup S$, i.e. we will show that \mathfrak{A}' is isomorphic to an \mathfrak{A}'' with unit $V \cup S$. (We will do this by putting z_0, z_1, \dots, z_N into the representation, one by one. In each step we will use an axiom $Ax_{k\ell}$ to tell us “where to put” z_i .) By an induction then \mathfrak{A} is isomorphic to an $\mathfrak{A}'' \in D_n$ with unit V such that all permuted versions of all repetition-free members of V are also in V . Then V is a G_n -unit by (*) and we will be done.

We may assume that \mathfrak{A} is atomic, because of the following. An equation in the language of Bo_n 's is called *positive in the wider sense* if complementation $-$ occurs only in front of some constant terms or in form of Boolean meet \cdot . Every Boolean algebra with operators \mathfrak{B} can be embedded into an atomic one \mathfrak{B}' such that all the equations valid in \mathfrak{B} which are positive in the wider sense continue to hold in the atomic one, \mathfrak{B}' , by [12, 2.15, 2.18] (see also [8, 2.7.5, 2.7.13]). All the axioms listed in this paper as well as the axioms defining ID_n in Bo_n given in [6] are positive in the wider sense. Any representation for \mathfrak{B}' gives a representation for $\mathfrak{B} \subseteq \mathfrak{B}'$ with the same unit. Thus we may assume that \mathfrak{A} is atomic.

A cylindric-relativised set algebra $\mathfrak{B} \in Crs_n$ is called *completely represented* if every sequence in the unit of \mathfrak{B} is in some atom, i.e. if the unit of \mathfrak{B} is the union – and not only the supremum – of the atoms of \mathfrak{B} . We also may assume that \mathfrak{A} is completely represented, because of the following. The representation for $\mathfrak{A} \in ID_n$ given on p.674 in [6] is a complete one, and in the above outlined induction step we obtain a complete representation from a complete one.

We say that an atom of \mathfrak{A} is *repetition-free* if it is below no d_{ij} (i.e. if it is below $\prod_{i < j < n} -d_{ij}$). Let a be an arbitrary repetition-free atom of \mathfrak{A} and let $i, j < n$, $i \neq j$.

Let now $n \geq 3$ and let

$$b \leq s_j^i c_j a \cdot s_i^j c_i a \cdot \prod_{k \neq i, j} s_i^k s_j^i s_k^j c_k a$$

be an atom. Such an atom exists by $\mathfrak{A} \models Ax$. We will show that b is repetition-free.

We will work in the *atom-structure* of \mathfrak{A} for a while. I.e. let At be the set of all atoms of \mathfrak{A} , let $E_{ij} = \{x \in At : x \leq d_{ij}\}$ and let $T_i = \{\langle x, y \rangle : x, y \in At, c_i x = c_i y\}$. Let $E_{ijk} = E_{ij} \cap E_{jk}$. We will freely use the following properties of atom-structures: For all $x, y \in At$

$$x \in E_{ij}, x T_k y \text{ imply } y \in E_{ij} \text{ if } k \neq i, j.$$

$$x, y \in E_{ij}, x T_i y \text{ imply } x = y.$$

$$E_{ij} \cap E_{jk} \subseteq E_{ik} = E_{ki}.$$

For any $x \in At$ and $i, j < n$, $i \neq j$ there is a unique $y \in At$ such that $x T_i y \in E_{ij}$.

It is not difficult to check that the first three statements hold in the atom-structure of any $\mathfrak{A} \in Crs_n$ while the last statement holds in the atom-structure of any $\mathfrak{A} \in D_n$.

First we show that $b \in E_{ij}$ would imply that $a \in E_{ij}$. Let $k < n$, $k \neq i, j$. Then $b \leq s_i^k s_j^i s_k^j c_k a$, and this means that there are atoms x, y, z of \mathfrak{A} such that $bT_k x, x \in E_{ik}, xT_i y, y \in E_{ij}, yT_j z, z \in E_{jk}, zT_k a$, see Figure 1.

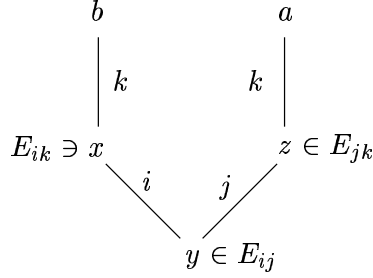


Figure 1:

Now, $b \in E_{ij}$ implies $x \in E_{ijk}$, which implies that $y = x$, $y \in E_{ijk}$ which imply that $z = y$, $z \in E_{ijk}$, which imply that $a \in E_{ij}$.

Next we show that $b \in E_{ik}$ would imply that $a \in E_{jk}$. By $b \leq s_i^j c_i a$ there is an atom x such that $bT_j x \in E_{ij}, xT_i a$, see Figure 2.

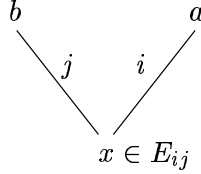


Figure 2:

Now $b \in E_{ik}$ implies $x \in E_{ijk}$ which implies that $a \in E_{jk}$.

Finally we show that $b \in E_{kl}$ would imply $a \in E_{kl}$ for all $k \neq \ell$, $j, i \neq k, \ell$. Let x be an atom as in Figure 2. Then $b \in E_{kl}$ implies that $x \in E_{kl}$ which implies that $a \in E_{kl}$.

Thus b is repetition-free, because a is repetition-free.

Assume now that $n \geq 2$ and $i, j \leq n$, $i \neq j$. For any repetition-free atom a choose a repetition-free atom b below $s_j^i c_j a \cdot s_i^j c_i a \cdot \prod_{k \neq i, j} s_i^k s_j^i s_k^j c_k a$ and let

$$f_{ij}(a) = b.$$

Such a b exists, because if $n \geq 3$, then we have seen this above, and if $n = 2$, then this holds by the axiom we required to hold.

Let now s be the repetition-free sequence we chose at the beginning of this proof, and recall the definition of $S = \{z_0, z_1, \dots, z_N\}$. We will define a sequence a_0, a_1, \dots, a_N of repetition-free atoms. Let a be an atom such that $s \in a$. Such an atom exists by our assumption that \mathfrak{A} is completely represented.

Let

$$a_0 \stackrel{\text{def}}{=} a.$$

Assume that a_ℓ has been defined for all $\ell < i$. Let $\ell < i$ and $m \neq j$ be such that $z_i = z_\ell \circ [m, j]$. (If there are several such ℓ, m, j , then we just select one such triple.) Now we define

$$a_i \stackrel{\text{def}}{=} f_{mj}(a_\ell).$$

We are ready to define the new representation of \mathfrak{A} : For any $x \in A$ define x' as

$$x' \stackrel{\text{def}}{=} x \cup \{z_i : a_i \leq x, i < N\}.$$

(Intuitively, this means that “we put the sequences z_i into the atoms a_i ”.)

Then $V' = V \cup S$, so the unit of the new representation will be $V \cup S$ as desired. We have to show that the function $x \mapsto x'$ is an embedding of \mathfrak{A} into the full (i.e. biggest) set algebra with unit $V \cup S$. Let us denote this function $x \mapsto x'$ by h .

Clearly, h is a one-to-one Boolean embedding, and $h(d_{ij}) = d_{ij}$ for all $i, j < n$. To show that h respects the cylindrifications c_k , first we prove an auxiliary statement.

Notation: If z is an n -sequence and $k, \ell < n$, then $z(k/z_\ell)$ denotes the sequence which agrees everywhere with z except on k , where it is the same as z_ℓ .

Lemma 2 *Assume that $z \in S$ and $z \in x'$, $x \in \text{At}$. Let $k, \ell < n$, $k \neq \ell$. Then for all $b \in \text{At}$ we have that*

$$(\star) \quad z(k/z_\ell) \in b \text{ iff } xT_k b \in E_{k\ell}.$$

Proof. By $z \in S$ there is an $m \leq N$ such that $z = z_m$. The proof will proceed by induction on m .

Assume that $m = 0$. Then $z_0 = s \in V$ and also $s \in x = a$ by $s \in x'$ and $s \in a = a_0$. Thus (\star) is true because \mathfrak{A} is a set algebra with D_n -unit V .

Assume that (\star) is true for all $k, \ell < n$ and for all $m < p$. We will show that (\star) is true for p , too. Let $z_p = z_m \circ [i, j]$ be such that $m < p$ and $a_p = f_{ij}(a_m)$. There are such m, i, j by the definition of a_p . Then $x = a_p$ by $z_p \in x'$, and also $z_m \in a_m$.

First we prove that $xT_k b \in E_{k\ell}$ imply that $z(k/z_\ell) \in b$.

Case 1 $k \neq i, j$ and $\ell = i$ or $\ell = j$. Assume first that $\ell = j$.

By $a_p = f_{ij}(a_m)$ we have that $a_p \leq s_i^k s_j^i s_k^j c_k(a_m)$. Since in D_n the so called Merry Go Round equation $s_i^k s_j^i s_k^j c_k x = s_j^k s_i^j s_k^i c_k x$ is true, then we also have that $x = a_p \leq s_j^k s_i^j s_k^i c_k(a_m)$. Thus there are atoms d, e, b^+ such that $xT_k b^+, b^+ \in E_{kj}, b^+ T_j e, e \in E_{ij}, e T_i d, d \in E_{ki}, d T_k a_m$. By $xT_k b \in E_{kj}$ then $b^+ = b$ (by the basic properties of atom-structures). See Figure 3.

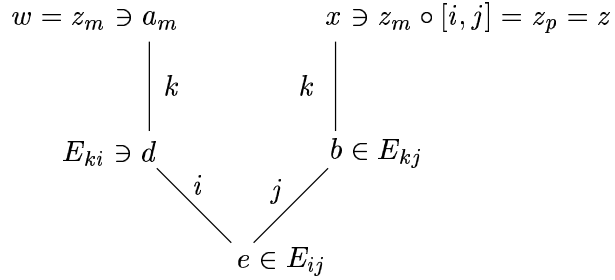


Figure 3:

Let $w = z_m$ and recall that $z = z_p$. Then $w(k/w_i) \in d$ by our induction hypothesis, and therefore $w(k/w_i)(i/w_j) \in e$ because \mathfrak{A} is a set algebra. But then $w(k/w_i)(i/w_j)(j/w_i) \in b$, again because \mathfrak{A} is a set algebra (notice that w_i is the k 'th member of the sequence $w(k/w_i)(i/w_j)$). Finally notice that $z(k/z_j) = w(k/w_i)(i/w_j)(j/w_i)$.

The case $\ell = i$ is completely similar, except that we do not have to use the Merry Go Round equation.

Case 2 $k \neq i, j$ and $\ell \neq i, j$.

Then $z(k/z_\ell) = z(k/z_j)(k/z_\ell)$ and therefore we will use the previous case. Let $b^+ \in E_{kj}$ be such that $xT_k b^+$. Such a b^+ exists by basic properties of atom-structures. Then by Case 1 we have that $z(k/z_j) \in b^+$, and then $z(k/z_j)(k/z_\ell) \in b$, because \mathfrak{A} is a set algebra, $z(k/z_j) \in V$ and $b^+ T_k b \in E_{k\ell}$.

Case 3 $k = i$ and $\ell = j$.

By $a_p \leq f_{ij}(a_m)$ we have that $a_m \leq s_j^i c_i(a_p)$, see Figure 4.

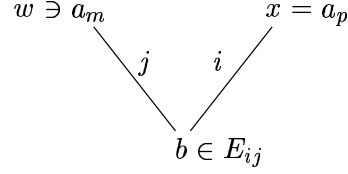


Figure 4:

Now, $w \in a_m$ implies, by the induction hypothesis, that $w(j/w_i) \in b$. But $w(j/w_i) = z(i/z_j)$.

The case $k = i$, $\ell \neq i, j$ is as above. The case $k = j$ is completely analogous.

Thus we have seen that $xT_k b \in E_{k\ell}$ imply that $z(k/z_\ell) \in b$. To see the other direction, assume that $z(k/z_\ell) \in b$. There is a b^+ such that $xT_k b^+ \in E_{k\ell}$, and we have seen that $z(k/z_\ell) \in b^+$ for this b^+ . Then $b = b^+$ because distinct atoms are disjoint from each other, and so $bT_k x$ since $b^+T_k x$. **QED(Lemma 2)**

Now we are ready to show that h is a homomorphism w.r.t. c_i . Let $j < n$, $j \neq i$. We will use that two sequences z and w differ only at i if and only if $z(i/z_j) = w(i/w_j)$.

To show that h is a homomorphism w.r.t. c_i amounts to showing (I)-(II) below for all atoms a, b and sequences $z, w \in V \cup S$:

- (I) $z \in a$, $w \in b$ and $z(i/z_j) = w(i/w_j)$ imply that $aT_i b$
- (II) $z \in a$ and $aT_i b$ imply that $z(i/z_j) = w(i/w_j)$ for some $w \in b$.

To prove (I)-(II), let a, b be two atoms, and let $aT_i a^+ \in E_{ij}$, $bT_i b^+ \in E_{ij}$. Then $z \in a$ implies $z(i/z_j) \in a^+$ if $z \in S$ by the previous lemma, and if $z \in V$, then this is so since \mathfrak{A} is a set algebra and $z(i/z_j) \in V$. Similarly, $w \in b$ implies $w(i/w_j) \in b^+$ for all $w \in V \cup S$.

Proof of (I): Assume $z \in a$, $w \in b$, $z(i/z_j) = w(i/w_j)$. Then $z(i/z_j) \in a^+$ and $w(i/w_j) \in b^+$ by the above, hence $a^+ = b^+$, which implies that $aT_i b$ by the definition of a^+, b^+ .

Proof of (II): Assume $z \in a$, $aT_i b$. Then $z(i/z_j) \in a^+$. Further, $aT_i b$ implies that $a^+ = b^+$. So $z(i/z_j) \in b^+$. Now since $b^+T_i b$, $z(i/z_j) \in V$, and since \mathfrak{A} is a set algebra, $z(i/z_j) \in b^+$ implies that $w(i/w_j) = z(i/z_j)$ for some $w \in b$.

By the above we have seen that h is an embedding of \mathfrak{A} into the full set algebra with unit $V \cup S$, i.e. \mathfrak{A} “can be represented on $V \cup S$.” Now repeating the above procedure along a transfinite induction we get a representation of \mathfrak{A} on $G(V) = \{s \circ \tau : s \in V \text{ and } \tau : n \rightarrow n\}$. Since $G(V)$ is a G_n -unit, this shows that $\mathfrak{A} \in IG_n$. By this, (i), (iii) of Theorem 1 are proved.

Let n be infinite and let Σ_n be the set of equations given in [6] which axiomatizes ID_n . We want to show that $\Sigma_n \cup Ax'$ axiomatizes the equational theory of G_n . We have already seen that $G_n \models \Sigma_n \cup Ax'$. Let now e be an arbitrary equation that holds in G_n . We want to show that e follows from $\Sigma_n \cup Ax'$. Let Γ be a finite subset of n , $|\Gamma| \geq 3$, which strictly contains all the indices occurring in e (i.e. Γ contains some other index, too). By [20, Lemma 4.13(ii)] then $G_\Gamma \models e$. Here G_Γ is the natural generalization of G_m to the case when m is any set of indices (and not necessarily an ordinal). Let $Ax_\Gamma \stackrel{\text{def}}{=} \{Ax_{ij\Gamma} : i, j \in \Gamma\}$. By Theorem 1(i) then $IG_\Gamma = \{\mathfrak{A} \in ID_\Gamma : \mathfrak{A} \models Ax_\Gamma\}$ since Γ is finite, $|\Gamma| \geq 3$. Let Σ_Γ be the set of axioms given in [6] which axiomatizes ID_Γ . Thus $\Sigma_\Gamma \cup Ax_\Gamma \models e$ by $G_\Gamma \models e$. By looking into [6], we see that $\Sigma_\Gamma \subseteq \Sigma_n$ by $\Gamma \subseteq n$. Also, $Ax_\Gamma \subseteq Ax'$. Hence $\Sigma_n \cup Ax' \models e$, this is what we wanted to show. We have seen that $\Sigma_n \cup Ax'$ axiomatizes the equational theory of G_n . Since IG_n is closed under subalgebras and direct products, the variety generated by G_n is HG_n . Thus $\mathfrak{A} \in HG_n$ iff $\mathfrak{A} \models \Sigma_n \cup Ax'$ iff ($\mathfrak{A} \in ID_n$ and $\mathfrak{A} \models Ax'$). This proves (ii) of Theorem 1. **QED(Theorem 1)**

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Alfréd Rényi Institute of Mathematics
 Budapest, Pf. 127
 H-1364, Hungary
 e-mail: andreka@math-inst.hu