of the Mal'cev's problem of the existence of an associative (semigroup) ring which is not embeddable into any skew-field, but its multiplicative semigroup is embeddable into a group.

For more informations, see also (Book and Otto 1993).

G.8 Relational Algebras

by Hajnal Andréka, Judit X. Madarász, and István Németi
in Budapest, Hungary

Boolean algebras (BA's for short) can be regarded as algebras of unary relations; i.e., the elements of a BA, say $B$, are unary relations and the operations of $B$ are the natural operations on unary relations. The purpose of relational algebra is to expand the natural algebras of unary relations (i.e., BA's) to natural algebras of relations of higher ranks, i.e., of relations in general. What will be the elements of the new algebras? The elements of BA's can be visualized as sets of points. Then, the elements of the new algebras will be sets of sequences (the reason for this is that the elements of relations are sequences independently of whether our relations are binary, ternary or $n$-ary).

The simplest case is when we concentrate on binary relations. For a set $U$, $\mathcal{P}(U)$ denotes its power set (the set of all subsets of $U$) while $\mathcal{P}(U)$ denotes the BA $(\mathcal{P}(U); \cup, \cap, -)$ with universe $\mathcal{P}(U)$. The full relation algebra over the set $U$ is defined to be the algebra

$$\mathcal{R}e(U) = (\mathcal{P}(U \times U), \circ, -1, \text{Id}_U)$$

where "$\circ$" is the usual composition of two relations, $R^{-1}$ is the usual converse (or inverse) of the relation $R$ and $\text{Id} = \text{Id}_U$ is the identity relation on $U$. The class $\mathcal{RRA}$ of representable relation algebras is defined as

$$\mathcal{RRA} = \mathcal{S}\mathcal{P}\{ \mathcal{R}e(U) \mid U \text{ is a set } \}$$

where $\mathcal{S}$ and $\mathcal{P}$ are the operators on classes of algebras corresponding to taking isomorphic copies of subalgebras and direct products, respectively.

G.8.1 Theorem (Tarski) $\mathcal{RRA}$ is a discriminator variety. The equational theory of $\mathcal{RRA}$ is recursively enumerable but not decidable.

Before discussing $\mathcal{RRA}$'s further, let us look at algebras of relations of higher ranks (e.g., ternary, $n$-ary relations). The natural algebras are
G.8 Relational Algebras
called cylindric algebras. In the following, \( n \) denotes a natural number.
The full cylindric algebra of \( n \)-ary relations over a set \( U \) is defined as
\[
\operatorname{Rel}_n(U) = (\mathcal{P}(U^n), c_0, \ldots, c_{n-1}, \text{Id})
\]
where \( \text{Id} \) is the \( n \)-ary identity relation \( \text{Id}_{n,U} = \{(a, \ldots, a) \mid a \in U\} \)
and \( c_i \) is a unary operation for each \( i < n \) defined by
\[
c_i(R) = c_i^U(R) = \{(b_0, \ldots, b_{i-1}, a, b_{i+1}, \ldots, b_{n-1}) \mid (b_0, \ldots, b_{n-1}) \in R \text{ and } a \in U\},
\]
for any \( i < n \) and \( R \subseteq U^n \). We will omit the superscript \( U \). Let \( R \subseteq U^n \) be a relation. Then the relation \( c_i(R) \) is called the smallest \( i \)-cylinder containing \( R \). Choosing \( n = 3 \) and \( U \) the real numbers, we obtain the greatest element \( U \times U \times U \) of our algebra as the usual Cartesian space, and \( i \)-cylinders appear as cylinders parallel to the \( i \)-th axis. Let \( n = 2 \) and \( R \subseteq U \times U \). Then \( c_0(R) = U \times \operatorname{Rg}(R) \) and \( c_1(R) = \operatorname{Dom}(R) \times U \). This example shows that the operations \( c_i \) are natural ones (on relations). The class \( \text{RCA}_n \) of \( n \)-ary representable cylindric algebras is defined as
\[
\text{RCA}_n = \mathbb{S} \mathbb{P}\{ \operatorname{Rel}_n(U) \mid U \text{ is a set } \}.
\]
G.8.2 Theorem (Tarski) \( \text{RCA}_n \) is a discriminator variety. The equational theory of \( \text{RCA}_n \) is recursively enumerable, and if \( n > 2 \) then undecidable.

To have all finitary relations over \( U \) in a single algebra, we need to extend cylindric algebras to \( \alpha \)-ary relations with \( \alpha \) an arbitrary ordinal. For this, we need to replace our single (\( \alpha \)-ary) identity relation \( \text{Id} \) with \( \alpha \times \alpha \) many identity relations \( \text{Id}_{ij} = \{ q \in U^\alpha \mid q_i = q_j \} \), for \( i, j \in \alpha \). Throughout, \( \alpha \) is an arbitrary (possibly finite) ordinal. Now, we define the full algebra of \( \alpha \)-ary relations as
\[
\operatorname{Rel}_\alpha(U) = (\mathcal{P}(U^\alpha), c_i, \text{Id}_{ij} \mid i, j < \alpha),
\]
where \( c_i(R) \) and \( \text{Id}_{ij} \) are defined as above. Thus, besides the Boolean operations, \( \operatorname{Rel}_\alpha(U) \) has \( \alpha \) many unary operations \( c_i \) (one for each \( i < \alpha \)) and \( \alpha \times \alpha \) many constants \( \text{Id}_{ij} \). Now, for \( \alpha < \omega \) we have two versions for \( \operatorname{Rel}_\alpha(U) \) but they are polynomially equivalent. Indeed, if e.g., \( \alpha = 3 \) then \( \text{Id}_{1,2} = c_0(\text{Id}) \) while \( \text{Id} = \text{Id}_{01} \cap \text{Id}_{12} \), \( \text{RCA}_\alpha = \mathbb{S} \mathbb{P}\{ \operatorname{Rel}_\alpha(U) \mid U \text{ is a set } \} \).

G.8.3 Theorem (Tarski) \( \text{RCA}_\alpha \) is an arithmetical variety. The equational theory of \( \text{RCA}_\alpha \) is recursively enumerable, but it is undecidable if \( \alpha > 2 \).
So far, the greatest elements of our algebras were Cartesian spaces, i.e., of the form $U^\alpha$ (both in the cases of RRA's and RCA's). However, this restriction is not always convenient (cf. e.g., Andréka et al. (1998), van Benthem (1996), Monk (2000), Henkin et al. (1981)). Removing this restriction motivates the definition of cylindric-relativized set algebras. Let $V \subseteq U^\alpha$ be an arbitrary $\alpha$-ary relation. Then the algebra of subrelations of $V$ is defined as

$$\text{Rel}(V) = (\mathcal{P}(V), c_i^V, Id_i^V \mid i, j < \alpha)$$

where $c_i^V(R) = V \cap c_i(R)$ and $Id_i^V = V \cap Id_i$. The class of $\alpha$-ary cylindric-relativized set algebras is defined as

$$\text{Crs}_\alpha = \{ \text{Rel}(V) \mid V \subseteq U^\alpha \text{ for some set } U \}.$$

The finite algebra part of the next theorem is the result of a joint work with Hajnal Andréka and Ian Hodkinson.

G.8.4 Theorem (Németi) Let $\alpha \neq 1$. Then $\text{Crs}_\alpha$ is an arithmetical variety. The equational theory of $\text{Crs}_\alpha$ is decidable. A finite $\text{Crs}_\alpha$ is isomorphic to one with finite greatest element.

It is natural to ask whether any one of the distinguished kinds RRA, RCA, $\text{Crs}_\alpha$ of algebras of relations is axiomatizable by a finite set of equations. If $\alpha \geq \omega$, then having a finite set of axioms is impossible because there are infinitely many basic operations, but we still could hope for a finite scheme of equations like the scheme $c_i Id_{i,j} = 1$, for all $i, j < \alpha$.

G.8.5 Theorem (Monk, Monk, Németi, Jónsson, Andréka)
Assume $\alpha > 2$. None of the varieties RRA, RCA, $\text{Crs}_\alpha$ is axiomatizable by a finite scheme of equations. None of RRA or RCA is axiomatizable by a scheme $\Sigma$ of universally quantified formulas such that $\Sigma$ involves only finitely many variables.

The negative result above motivates the definition of the finitely axiomatizable approximations RA and CA of RRA and RCA. The axioms for RA are (R1) – (R3) below.

(R1) The Boolean axioms; and the operations $\circ, ^{-1}$ are "$\cup$"-distributive, i.e., they commute with the Boolean join "$\cup$".
(R2) $\circ$, $^-1$, $\text{Id}$ form an involuted monoid, where an \textit{involuted monoid} is a monoid with an extra unary operation $^-1$ satisfying the two equations $x^{-1-1} = x^{-1}$, $(x \circ y)^{-1} = y^{-1} \circ x^{-1}$.

(R3) $x^{-1} \circ -(x \circ y) \leq -y$.

The axioms for $\text{CA}_\alpha$ are (E1) - (E5) below.

(E1) The Boolean axioms.

(E2) The $c_i$'s are commuting complemented $\cup$-distributive closure operations (e.g., $c_i c_j x = c_j c_i x$, $c_i c_j x = -c_i x$ etc).

(E3) $\text{Id}_{ii} = 1$ and $\text{Id}_{ij} = \text{Id}_{ji}$ (i.e., notational trivialities)

(E4) $\text{Id}_{ik} = c_j (\text{Id}_{ij} \cap \text{Id}_{jk})$ if $j \notin \{i, k\}$.

(E5) $x \leq \text{Id}_{ij} \Rightarrow c_i (x) \cap \text{Id}_{ij} = x$.

Clearly, $\text{RA} \supseteq \text{RRA}$ and $\text{CA}_\alpha \supseteq \text{RCA}_\alpha$. $\text{CA}_1$'s are also called \textit{monadic algebras}. Both approximations $\text{RA}$ and $\text{CA}_\alpha$ were introduced by Tarski. In some sense, $\text{RA}$ is close to $\text{RRA}$ and $\text{CA}_\alpha$ is close to $\text{RCA}_\alpha$. However, it is hard to make it precise what we mean by \textit{close} here. It is possible to introduce natural properties such that

$$\text{RA} \cap \text{"property" } \subseteq \text{RRA} \text{ and } \text{CA}_\alpha \cap \text{"property" } \subseteq \text{RCA}_\alpha.$$  

However, one can replace $\text{RA}$ by a bigger class $\text{RA}^-$ and $\text{CA}_\alpha$ with $\text{CA}_-\alpha$ such that all the above style representation theorems remain true. Using the well established connections between logic and algebraic logic, one can argue that the axioms for $\text{RA}$ and $\text{CA}_\alpha$ are optimal in some sense. E.g., the $\text{CA}_\alpha$ axioms correspond to one of the usual axiomatizations of first order logic and most of the equations separating $\text{RCA}_\alpha$ from $\text{CA}_\alpha$ would look strange to the logician as a possible extra axiom (unless he is trying to axiomatize the finite-variable fragments $L_n$ of first order logic).

$\text{CA}_\alpha$'s correspond to first order logic $L_\alpha$ with equality. If we \textit{algebraize} the same logic, but without equality, we obtain \textit{substitution-cylindrification algebras} which are obtained from $\text{CA}_\alpha$ by throwing away the constants $\text{Id}_{ij}$ and replacing them with the term-functions $s^i_j (x) = c_i (\text{Id}_{ij} \cap x)$. \textit{Quasi-polyadic algebras} (of Halmos) are almost the same as these, cf. Henkin et al. (1985) for both kinds of algebras.

The negative result Theorem G.8.5 gave rise to the Finitization Problem which asks whether we could define our algebras of relations in such a way that they would form a finitely axiomatizable variety. There has been extensive research work on this problem recently, cf. e.g., Németi and Sain (2000) for further references.

Algebras of relations have been extensively applied in computer science, AI, linguistics and other areas, cf. e.g., Bergman et al. (1990), Marx et al. (1996), van Benthem (1996).

This research was supported by the Hungarian Foundation for Basic Research, Grants T30314 and T35192.

G.9 Partial Algebras

by Peter Burmeister in Darmstadt, Germany

Introduction

Quite often (e.g., in Computer Science, but also for the multiplicative inverse in fields) the operations in an "algebra" are not everywhere defined. Moreover, quite often constructions for "total algebras" make use of partial algebras and some general construction principles like universal solutions. For such "partial algebras", a highly developed theory has been worked out. This theory of partial algebras lies in between those of total algebras and relational systems (see Section G.8). From total algebras it inherits in particular the concepts of terms, direct products (with the structure defined componentwise whenever possible in all components), closed subsets (and in connection with them (closed) subalgebras as the partial algebras obtained by restricting the structure to closed subsets, and the concept of generation). From relational systems it inherits the wealth of possible concepts, since partial algebras can model relational systems (cf. Burmeister 1986, 13.4.2). Moreover, many-sorted (partial) algebras can easily be considered as partial algebras on the disjoint union of the carriers of the different sorts, and their homomorphisms then have just to be compatible with the canonical homomorphisms into the set of sorts with the specification of the many-sorted (partial) operations as fundamental partial operations (cf. Burmeister 1986). Here we can only introduce some of the basic concepts and applications of a language for partial algebras.
The Concise Handbook of Algebra

by

Alexander V. Mikhalev
Department of Mechanics and Mathematics, Moscow State University, Moscow, Russia

and

Günter F. Pilz
Institute of Algebra, Johannes Kepler University-Linz, Linz, Austria

KLUWER ACADEMIC PUBLISHERS
DORDRECHT / BOSTON / LONDON