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ON A GENERALIZATION OF THE GAME GO-MOKU I

by

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Abstract

We investigate the winning and drawing strategies in the generalized Go-moku games which are defined in 1.1. It is proved that every open game is equivalent, in a certain sense, to some generalized Go-moku. We give an example of a recursive game in which player I has a winning strategy but has no recursively enumerable one. Examples are constructed for non-determined games of length $\omega+20$ and $\omega\cdot 2$ under certain set-theoretical assumptions. Finally, we determine the possible minimal lengths of winning strategies.

0. Introduction

The well-known game Go-moku [5] is played on an infinite chessboard. Two players, I and II, occupy the squares alternately. The winner is the player who has first 5 or more adjacent squares in a row, horizontally, vertically or along either diagonal. An easy argument shows (see, e.g., Proposition 2.1 below) that II cannot have a winning strategy (WS in the sequel) and either I has a WS or II has a drawing strategy (DS), i.e., a strategy which allows II to play indefinitely.

As far as our knowledge goes, it is not known which is the case in Go-moku. It is not even known whether a WS of I, if any, can be bounded in time. In other words, assume that I has a WS. Is there a natural number n such that I can win before his n-th move? The best available result is in [6, pp. 257—258]:

THEOREM 0.1. *If the game Go-moku is played on countably many boards (i.e., at every step the player chooses a board and occupies a square on it) and* I *has a WS then* II *does not have arbitrarily long counterplay. (A WS can always be bounded in time.)* ∎

The game n-Go-moku is a slight modification of the original game. The winner is required to have at least n adjacent squares. It is an easy exercise to prove the

PROPOSITION 0.2. I *wins* 4-*Go-moku in at most* 6 *moves.* ∎

On the other hand we succeeded in proving (cf. [1])

THEOREM 0.3. *For $n>8$,* I *has no WS in n-Go-moku.*

PROOF (Sketch). We describe a strategy of II which prevents I from occupying 9 adjacent squares. Divide the board into pieces of size 5×5 along the border-lines of the squares. Let II play independently in each of these pieces. (If there is no more room for his move, occupy a square in any other piece.) The goal of II in each of these pieces is to prevent I from occupying either a full row, a full column, or one of the ten diagonals shown in Fig. 1. It can be checked that II can achieve his goal. I cannot have 9 or more adjacent squares because their intersection with some piece would be a part which I was prevented from occupying. ∎

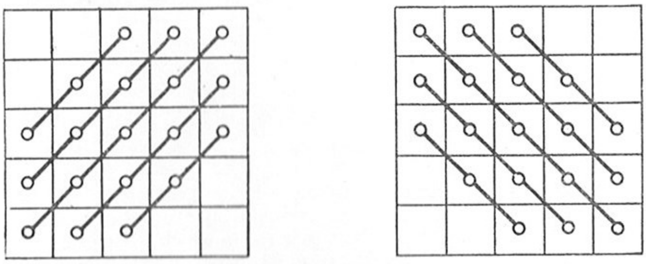

Fig. 1

We know nothing about the missing cases $5 \leqq n \leqq 8$[1].

These results led to some generalizations of the game Go-moku, one of which is discussed in detail here. For other generalizations, see our forthcoming paper.

1. Definitions

Our set-theoretical notation will be standard. Ordinal numbers are denoted by α, β, etc., ω denotes the first infinite ordinal as well as the cardinality of countable sets. The cardinality of the continuum is denoted by 2^{ω}. Sequences are enclosed between angular brackets $\langle$ and $\rangle$, for example the empty sequence is denoted by $\langle\rangle$.

The family of all finite 0—1 sequences is denoted by ${}^{\underset{\smile}{\omega}}2$, and ${}^{\omega}2$ is the family of 0—1 sequences of length ω. If $\sigma \in {}^{\omega}2$ then $\sigma | n \in {}^{\underset{\smile}{\omega}}2$ is the unique initial segment of σ of length n. Let s and t be sequences; the relation $s \prec t$ holds if s is a proper initial segment of t.

1.1. $\mathfrak{A}$-*games*

The $\mathfrak{A}$-game $[A, F]^{\alpha}$ consists of the board A, the family F of nonempty finite subsets of A which are the winning sets, and the ordinal number α which is an upper bound for the length of the game.

$\mathfrak{A}$-games are played by two players, I and II. They occupy elements of A alternately. Every element can be chosen at most once. I begins and every limit step (if any) is I's turn. The game ends if either I or II occupies all elements of some $X \in F$ (covers X), the winner is the one who does it. The game ends if all elements of A have been chosen or if α moves have been made. In these cases the game is a draw. If $\alpha = \omega$, the game is said to be finite.

[1] R. K. GUY kindly informed us that Andreas BROUWER and others can prove Theorem 0.3 for $n=7$ and $n=8$.

1.2. $\mathfrak{B}$-*games*

The $\mathfrak{B}$-game $[A, F^0, F^1]^\alpha$ consists of A and α as above, and of the families F^0 and F^1 of finite subsets of A. $\mathfrak{B}$-games are played by I and II as follows. First I chooses an element of $\{0, 1\}$ which we denote by k. After this they occupy elements of A alternately, starting with I if $k=0$, and with II if $k=1$. The title of the player who starts picking is W, the other's is B (from white and black). Every limit step (if any) is I's turn. The game ends if either I covers some element of F^k, or II covers some element of F^{1-k}; the winner is the one who does it. The game ends if all elements of A have been chosen or if α moves have been made. In these cases the game is a draw. If $\alpha=\omega$, the game is said to be finite.

1.3. *Positional games*

Positional games contain (at least an evidently equivalent form of) every infinite two-person game of perfect information of length ω where the families of the winning positions of both players are open sets [4]. The components of a positional game are a sequence of sets $\langle A_i\colon i\in\omega\rangle$, and two disjoint sets F_W and F_B of finite sequences such that $\langle a_0, a_1, \ldots, a_k\rangle\in F_W\cup F_B$ implies $a_i\in A_i$ $(i\leqq k)$. We assume that no sequence in $F_W\cup F_B$ is a proper initial segment of any other element.

Two players, W and B take turns alternately. First W picks $a_0\in A_0$ then B picks $a_1\in A_1$, again W picks $a_2\in A_2$, etc. W wins if $\langle a_0, a_1, \ldots, a_k\rangle\in F_W$ for some k, B wins if $\langle a_0, a_1, \ldots, a_k\rangle\in F_B$ for some $k\in\omega$, otherwise the game is a draw.

Evidently, every finite $\mathfrak{A}$ and $\mathfrak{B}$-game can be transformed into a positional game. This transition can be done by a recursive function.

1.4. *Snub-games*

Let G be any positional game. The snub-G snub-game is played by I and II as follows. First I chooses who he wants be: W or B. If he chooses W, he begins the game G as W and II plays as B. Otherwise II begins the game as W and I plays as B. The other moves go by the rules of G. The winner of the game G is the winner of the game snub-G.

1.5. *Strategies, equivalent games*

The notion of strategy and that of play according to a strategy is discussed in [8]. The strategy S is a I-winning strategy, I-WS in short, if every play according to S is a win for I. The strategy S is a I-drawing strategy, I-DS, if every play according to S is either a win for I or is a draw. Similarly, for II-WS, II-DS, etc.

The winning strategy S is *α-bounded*, if there is a $\gamma<\alpha$ such that every play according to S ends before the γ-th move. In case of $\mathfrak{A}$- and $\mathfrak{B}$-games, the I-winning strategy S is *bounded in space*, if there is a finite subset of the board such that in every play according to S, I occupies elements of this subset only. Hence boundedness in space implies ω-boundedness.

A game is *determined* if either both players have DS or one of them has WS.

Let the games G_0, and G_1 be played, say, by I and II. We say that G_0 is *equivalent* to G_1, if there are recursive functions φ_0 and φ_1 (in the sense of [3]) such that

(i) $\varphi_i(G_i)=G_{1-i}$, $(i=0, 1)$;

(ii) if S is I-WS (I-DS, II-WS, II-DS) in G_i then $\varphi_i(S)$ is a I-WS (I-DS, II-WS, II-DS) in G_{1-i} $(i=0, 1)$.

Obviously, every finite $\mathfrak{A}$- and $\mathfrak{B}$-game is equivalent to some positional game. Equivalent games are equidetermined.

2. Basic results

PROPOSITION 2.1. *In $\mathfrak{A}$-games* II *has no WS.*

PROOF. Suppose the contrary and let S be a II-WS. Let I play by S as follows. Choose any $t_0 \in A$ (an "imagined" element) and let I's first move be the answer by S to this imagined move. If II's answer is not the imagined element, answer I simply by S. If it is, drop t_0 as an imagined element, choose an entirely new one and regard it as the move of the other. At limit steps I always has to choose a new imagined element. Now just the strategy ensures I to cover a winning set before II could do it, which is a contradiction. ∎

The same argument as above shows that

PROPOSITION 2.2. *In $\mathfrak{A}$-games, $\mathfrak{B}$-games, and snub-games* II *has no WS. If* II *has DS then* I *has DS, too.* ∎

REMARK. This proposition remains true even if we allow the winning sets to be infinite.

PROPOSITION 2.3. *Positional games are determined.*

PROOF. This is the GALE—STEWART result for open games [4]. If I has no WS, II make a move such that I still has no WS. Since if I wins he wins after finitely many moves, this strategy is a DS for II. ∎

COROLLARY 2.4. *Finite $\mathfrak{A}$ and $\mathfrak{B}$-games as well as snub-games are determined. In these games* I *always has DS.* ∎

PROPOSITION 2.5. *Even in the case of finite $\mathfrak{A}$-games, the existence of a* I-*WS does not imply the existence of an ω-bounded WS; the existence of an ω-bounded* I-*WS does not imply the existence of a space-bounded WS.*

PROOF. We give two examples which witness the assertions. Let first B_1 and B_n for $n \geqq 2$ be the set of nodes of the trees shown in Fig. 2.

Let $F_i=\{$the full branches of $B_i\}$, for example, every element of F_1 consists of four nodes. I has WS in $[\bigcup_{i\geqq 1} B_i, \bigcup_{i\geqq 1} F_i]^\omega$ because I can win in B_1, but II may postpone his defeat for n moves by threatening in B_n.

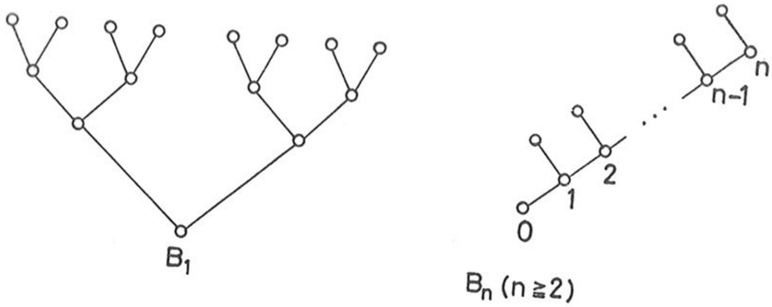

Fig. 2

In the second example let the board be $A=\{R\}\cup\{P_j^i\colon\ i\in\omega,\ j\leqq 7\}$, and let $A_k=\{R\}\cup\{P_j^i\colon\ i<k,\ j\leqq 7\}$. The winning sets are

$$\{R, P_0^i, P_1^i\},\ \{R, P_0^i, P_2^i, P_3^i\},\ \{R, P_0^i, P_2^i, P_4^i, P_5^i\},\ \{R, P_0^i, P_2^i, P_4^i, P_6^i\}$$

and

$$\{P_1^i, P_3^i, P_5^i, P_7^j\} \quad \text{for all} \quad i\leqq j\in\omega$$

(see Fig. 3). I wins this game playing as follows. Start with occupying R. The response of II is an element of some A_k, then picks $P_0^k, P_2^k, P_4^k, P_6^k$ in succession. On the other hand II has counterplay in every A_k picking first either R or P_7^{k-1}. ∎

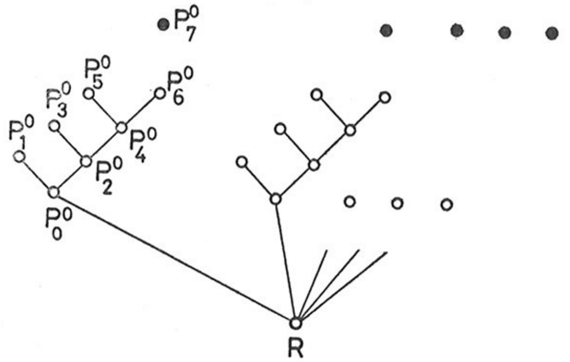

Fig. 3

We define the *rank* of I-WS of finite $\mathfrak{A}$-games as follows. Winning strategies can be regarded as trees, the root is I's first move, the edges starting from the root are labelled by the possible moves of II, the nodes at the other end are I's responses by the strategy, etc. To each node v assign the least ordinal which is greater than the ordinals assigned to the successors of v. This definition is sound because these trees are well-founded. The ordinal assigned to the root is the rank of the strategy. For example a I-WS is ω-bounded if and only if its rank is less than ω.

The rank of a finite $\mathfrak{A}$-game is the infimum of the ranks of I-winning strategies. So the rank of the first example in Proposition 2.5 is ω. The construction described in the proof of Theorem 4.6 gives

PROPOSITION 2.6. *For every ordinal α there is a finite $\mathfrak{A}$-game of rank* $\geqq \alpha$. ▮

3. Equivalence of finite games

In Section 1 we have remarked that every finite $\mathfrak{A}$-game is equivalent to some positional game. The converse cannot be true because there are positional games where II has WS. But in view of 2.4 we may hope for

THEOREM 3.1. *Every snub-game is equivalent to some finite $\mathfrak{A}$-game.*

PROOF. The theorem is an immediate consequence of Lemmas 3.7 and 3.8 below. ▮

To describe the construction we shall need the following structure.

DEFINITION 3.2. Let Γ be an index set. The *broom* associated with Γ is the 7-tuple $\langle B, B^0, B^1, C^0, C^1, D^0, D^1\rangle$ where B is a set, C^0 and C^1 are functions from Γ to finite subsets of B, and the others are families of finite subsets of B.

The set B consists of the points of the handle $H=\{U, V_0, V_1, V_2\}$ and of the points of the broomcorn $J=\{X_j^a\colon j\leqq 8,\ a\in\Gamma\}$, see Fig. 4. The elements of $B^0\cup B^1$ are the *winning* sets, B^0 consists of the subsets $\{X_0^a, X_1^a\}$, $\{X_1^a, X_2^a, X_3^a\}$, $\{X_1^a, X_3^a, X_4^a, X_5^a\}$, $\{X_1^a, X_3^a, X_5^a, X_6^a, X_7^a\}$ and $\{X_1^a, X_3^a, X_5^a, X_7^a, X_8^a, U\}$ for all $a\in\Gamma$, and B^1 contains all two-element subsets of H and all 8-element subsets of J. The values of functions C^0 and C^1 are the *choosing* sets $C^0(a)=\{X_1^a, X_3^a, X_5^a, X_7^a, U\}$ and $C^1(a)=\{X_0^a, X_2^a, X_4^a, X_6^a, X_8^a\}$ for all $a\in\Gamma$. Finally, the elements of D^0 and D^1 are the *validating* sets, D^0 contains the two-element subsets $\{U, V_0\}$, $\{U, V_1\}$, $\{U, V_2\}$, and D^1 contains the one-element subsets $\{V_0\}$, $\{V_1\}$, $\{V_2\}$. ▮

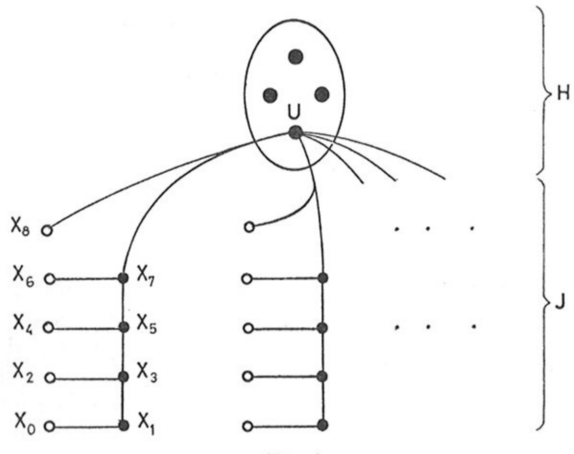

Fig. 4

LEMMA 3.3. *Let $\langle B, B^0, B^1, C^0, C^1, D^0, D^1\rangle$ be the broom associated with Γ. The $\mathfrak{B}$-game $[B, B^0, B^1]^\omega$ has the following property. Either one of the players can win, or they cover the choosing sets $C^0(a)$, $C^1(a)$ for exactly one $a\in\Gamma$ and W covers a validating set from D^0 in the first 11 steps. In the latter case B has a two-step threat playing which he can cover a validating set from D^1. Except for this, no player has a realizable two-step threat.*

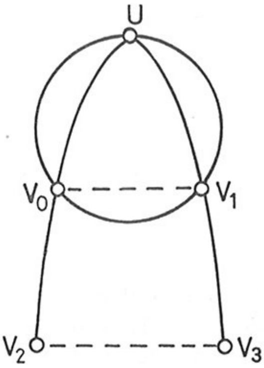

Fig. 5

PROOF. B threatens a two-step victory in the handle so W always has to counterthreaten. Therefore W must pick X_1^a, X_3^a, X_5^a, X_7^a, U and one of V_i's in succession for some $a\in\Gamma$, because W cannot counterthreaten more than 8 times. Then B picks a free element of the handle which forces W to pick the remaining point.

After these steps B has no more two-step threat whilst W has a lot. But W cannot cover more choosing set and cannot threaten more than three times because then W loses the game. ∎

A simplified version of brooms for the case $\Gamma=\{0, 1\}$ is the brush as follows.

DEFINITION 3.4. A *brush* is the 7-tuple $\langle B, B^0, B^1, C^0, C^1, D^0, D^1\rangle$. The set B consists of 5 points, U and V_j ($j\leqq 3$), see Fig. 5. The family B^0 consists of $\{U, V_0, V_1\}$, $\{U, V_0, V_2\}$, and $\{U, V_1, V_3\}$. The family B^1 contains $\{U\}$, $\{V_0, V_1\}$, and $\{V_2, V_3\}$. The functions C^0 and C^1 are defined as $\mathrm{Dom}\,(C^0)=\mathrm{Dom}\,(C^1)=\{0, 1\}$ and

$$C^0(0) = C^1(1) = \{V_0, V_3\} \qquad C^0(1) = C^1(0) = \{V_1, V_2\}.$$

Finally, $D^0=D^1$ and they contain the subsets $\{V_0, V_3\}$ and $\{V_1, V_2\}$. ∎

The properties of a brush are the same as of a broom, furthermore

LEMMA 3.5. *After playing off a brush, all the points are occupied and no player has any threat.* ∎

We interpret playing off a broom as W's choosing exactly one $a\in\Gamma$, and playing off a brush as B's choosing either 0 or 1.

Brooms (and brushes as well) can be fitted together to form an ω-long sequence. The winning sets of the i-th member of the sequence are validated by the corresponding validating sets of the $(i-1)$st member. So the players are forced to play off these brooms in sequence, exchanging the role of W and B at every new broom. A bit more formally,

DEFINITION 3.6. Let $\langle B_i, B_i^0, B_i^1, C_i^0, C_i^1, D_i^0, D_i^1\rangle$ be brooms (brushes) associated with $\Gamma_i(=\{0, 1\})$ for $i\in\omega$ and let $D^0_{-1}=D^1_{-1}=\emptyset$. The ω-broom (ω-brush) is the triplet $\langle B, B^0, B^1\rangle$ such that for $k=0, 1$

$$B = \bigcup\{B_i\colon\ i\in\omega\}$$

$$B^k = \{X\cup Y\colon\ X\in D^k_{2i} \text{ and } Y\in B^{1-k}_{2i+1} \text{ or } X\in D^{1-k}_{2i-1} \text{ and } Y\in B^k_{2i},\ i\in\omega\}.$$ ∎

After these preliminaries we turn our attention to the proof of Theorem 3.1. We start by

LEMMA 3.7. *Every snub-game is equivalent to some finite $\mathfrak{B}$-game.*

PROOF. Let the positional game G be given by the sets A_i $(i\in\omega)$ and by the winning sets F_W and F_B. Let $\langle B, B^0, B^1\rangle$ be the ω-broom built from the brooms $\mathscr{B}_i=\langle B_i, B_i^0, B_i^1, C_i^0, C_i^1, D_i^0, D_i^1\rangle$ associated with the sets A_i. We are going to define a $\mathfrak{B}$-game G^* which is equivalent to the game snub-G. The board of G^* is B. For every finite sequence $s=\langle a_0, a_1, \dots, a_n\rangle$ such that $a_i\in A_i$ let

$$S_s^0 = C_0^0(a_0)\cup C_1^1(a_1)\cup\dots\cup C_n^k(a_n)$$
$$S_s^1 = C_0^1(a_0)\cup C_1^0(a_1)\cup\dots\cup C_n^{1-k}(a_n)$$

with $k=0$ if n is even and $k=1$ if n is odd. Now let

$$F^0 = B^0\cup\{S_s^0:\ s\in F_W\}$$
$$F^1 = B^1\cup\{S_s^1:\ s\in F_B\}.$$

We claim that the games snub-G and $G^*=[B, F^0, F^1]^\omega$ are equivalent. (i) of the definition is evidently true. (ii) follows from the fact that the players in G^* are forced to simulate the game G. I's choosing to be W in G corresponds to I's choosing the winning sets F^0 in G^* (i.e., being W in G^*, too). In G^* W is forced to start to play off the broom $\mathscr{B}_0$ (otherwise he loses the game). So first the players play off the broom $\mathscr{B}_0$ and player W covers some $C_0^0(a_0)$ with $a_0\in A_0$, i.e., W chooses this a_0. Moreover they validate the winning sets of the broom $\mathscr{B}_1$ by some validators in D_0^0 and D_0^1 so next they have to play off $\mathscr{B}_1$. Here player B covers some $C_1^0(a_1)$, i.e., B chooses $a_1\in A_1$, and so on. If any player does not follow this simulation, he loses.

Last, the winning sets of the form S_s^k ensure in G^* the victory to that player who wins in the snub-G game. ∎

LEMMA 3.8. *Every $\mathfrak{B}$-game of limit length is equivalent to some $\mathfrak{A}$-game of the same length. If β is a limit ordinal then β-bounded strategies are preserved.*

PROOF. Let the $\mathfrak{B}$-game be $G=[B_0, F_0^0, F_0^1]^\alpha$ and let $[B_1, F_1^0, F_1^1]^\alpha$ be a new instance of G. Our aim is to construct an $\mathfrak{A}$-game $G^*=[A, F]^\alpha$ in which the players are forced to simulate G. B_0 and B_1 will be subsets of A. The elements of F_k^0 (F_k^1) amplified with some validating sets are among the winning sets of G^*. These validating sets are subsets of the finite set $A-B_0-B_1$. We demand these sets to have the following property. Both players must be able either to win or to cover some validating set within finitely many moves. Moreover, if a player validates (the elements of) F_k^0, the other must not be the first occupying any element of B_k. If he validates F_k^1, the other must have no more than one occupied element in B_k.

This property ensures the portability of WS and DS from G^* to G and back. We describe here a structure satisfying this. It consists of 10 points, U_k and $V_{i,j,k}$ for $i, j, k=0, 1$. There are two F_k^0 validating sets (Fig. 6)

$$D_k^0 = \{U_0, V_{k,0,0}, V_{k,1,0}, V_{1-k,0,1}\}$$

and

$$D_k^1 = \{U_0, V_{k,0,0}, V_{k,1,0}, V_{1-k,1,1}\}$$

and there is one F_k^1 validating set

$$D_k^2 = \{U_1, V_{k,0,1}, V_{k,1,1}\}$$

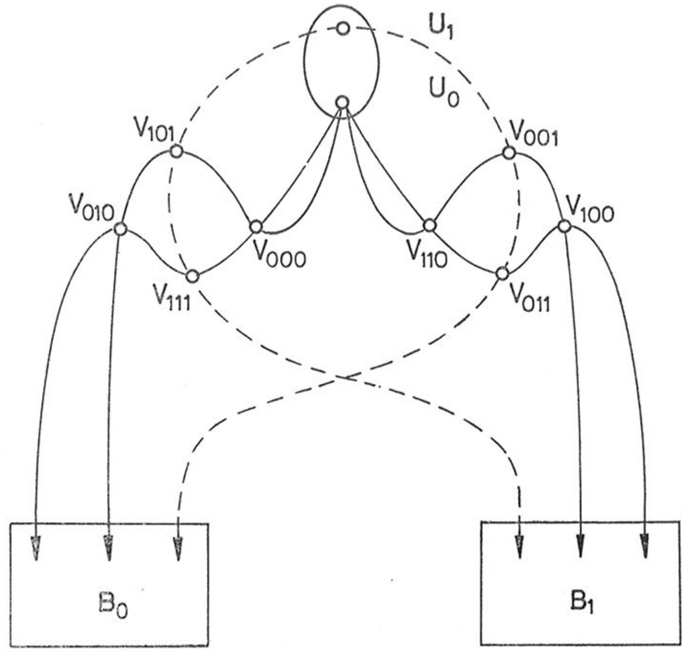

Fig. 6

for $k=0, 1$. The only two-element winning set in F is $\{U_0, U_1\}$. If I's first move is U_0 we interpret it as I's choice $k=0$ in the game G. Otherwise II's first move is U_0, therefore we may assume that in the first two moves they occupy U_0 and U_1. Other winning sets in F are

$$\{U_0, V_{i,j,0}, V_{i,j,1}\} \quad \text{for} \quad i, j = 0, 1$$

and those among the sets

$$\{U_1, V_{0,0,k_1}, V_{0,1,k_2}, V_{1,0,k_3}, V_{1,1,k_4}\}$$

which do not contain D_0^2 or D_1^2.

Let $A=\{U_k, V_{i,j,k}\}\cup B_0\cup B_1$ and let F consist of the 14 winning sets defined above and of the sets $D_k^0\cup X$, $D_k^1\cup X$, $D_k^2\cup Y$ with $X\in F_k^0$ and $Y\in F_k^1$ for $k=0, 1$.

It is easy to check that for $G^*=[A, F]^\alpha$ the (ii) of the definition of the equivalence holds and (i) is trivial. ∎

This proves Theorem 3.1, too.

The following lemma is about the converse of 3.8.

LEMMA 3.9. *Suppose that there is a recursively definable choice function on the board of the $\mathfrak{A}$-game $G=[A, F]^\alpha$. (This is the case if A is an ordinal number.) Then G is equivalent to the $\mathfrak{B}$-game $G^*=[A, F, F]^\alpha$. Moreover, β-bounded strategies are preserved for every ordinal β.*

PROOF. Both I and II can carry strategies recursively by the method of "imagined elements" as was described in 2.1 because there is a recursive choice function on A which gives imagined elements whenever they are necessary. ∎

We mention here a rather surprising consequence of these lemmas.

COROLLARY 3.10. *There is a finite* $\mathfrak{A}$*-game* $[\omega, F]^{\omega}$ *such that*

(i) *every element of* F *has fewer than* 100 *elements;*
(ii) F *is recursive, i.e., there is a recursive procedure which decides whether a given finite subset of* ω *is an element of* F *or not;*
(iii) *every play ends before the* 100*th move, no matter how the players play;*
(iv) *(as a consequence of* (iii)*) player* I *has WS, but*
(v) I *has no recursively enumerable WS.*

PROOF. We define a positional game G as follows. Let Φ be the set of all formulas of the ZF set-theory and let $\Psi\subset\Phi$ be the ZFC-provable formulas. Obviously, both Φ and Ψ are countable and, by Gödel's theorem, Ψ is not recursive. The game starts by W's saying 1 (if he does not say 1, he loses). Then B says a formula $\varphi\in\Phi$, W says a proof from ZFC, and finally B says a proof from ZFC, too. W wins if either he has proved φ or neither he nor B proved φ. Otherwise the winner is B.

Obviously, W has WS in G because if a formula is provable, he can prove it. On the other side I has no recursively enumerable WS. Supposing it were so, there would be a recursive function which assigns a proof to every provable formula, i.e., Φ would be recursive, a contradiction.

Because W has WS in G, W has WS but has no recursively enumerable WS in snub-G. Making the transformations as was described in 3.8 and 3.7 we get the desired game. ∎

4. Infinite games

In this section we study the determinacy and boundedness of $\mathfrak{A}$-games $[A, F]^{\alpha}$ with limit $\alpha>\omega$. If $|A|<\omega$ or $|F|<\omega$ then the game is equivalent to $[A, F]^{n}$ for some $n<\omega$ so we can assume $|A|\geqq\omega$ and $|F|\geqq\omega$. In these latter cases, however, the study of $\mathfrak{B}$-games gives all the information as was shown in Lemmas 3.8 and 3.9 because the cardinality of the boards and that of the families of winning sets are preserved.

First we recall here the basic properties of ω-brushes (see Definition 3.6).

LEMMA 4.1. *Suppose that the* $\mathfrak{B}$*-game* $[A, F^0, F^1]^{\alpha}$ *with* $\alpha\geqq\omega$ *contains the* ω*-brush* $\langle B, B^0, B^1\rangle$ *(i.e.,* $B\subset A$, $B^0\subset F^0$, $B^1\subset F^1$*) and there is no two-step threat but in the* ω*-brush. Then the players are forced to play off the elements of the* ω*-brush in the first* ω *moves so that*

(i) *after these moves every point of the* ω*-brush is occupied;*
(ii) *the choosing sets* $C_i^0(d_i)$, $C_i^1(d_i)$ $(d_i\in\{0, 1\})$ *are covered in succession;*
(iii) *the value of the digit* d_i *is chosen by* B *if* i *is even and by* W *if* i *is odd in possession of full information about the previous digits and no information about the rest of the digits.*

Let $s=\langle d_0, d_1, \dots, d_n\rangle$ *be any finite* 0—1 *sequence (i.e.,* $s\in{}^{\underset{\smile}{\omega}}2$*) and let*

$$(s)_W=\bigcup\{C_i^0(d_i)\colon\ i\leqq n\}\qquad (s)_B=\bigcup\{C_i^1(d_i)\colon\ i\leqq n\}.$$

In the ω-brush either one of the players may win before the ω-th step or there is exactly one $\sigma\in{}^{\omega}2$ such that every $(\sigma|n)_W$ is covered by W and every $(\sigma|n)_B$ is covered by B. ∎

This lemma says that an ω-brush forces the players to play an infinite 0--1 game [4].

THEOREM 4.2. *There is a non-determined $\mathfrak{A}$-game the board of which has cardinality at most 2^ω such that some player may win before the $(\omega+20)$th step.*

PROOF. By the remarks at the beginning of this section, it is enough to give a $\mathfrak{B}$-game with these properties. The non-determinacy means, by Proposition 2.2, that I has no DS, i.e., if either player W or player B plays by a strategy, he loses.

Now it is well-known that there are non-determined 0—1 games [4], let the family of sequences $C\subset{}^{\omega}2$ witness it. Of course, the cardinality of C is $\leqq 2^\omega$. We build a $\mathfrak{B}$-game G as follows. We start with an ω-brush. If the players in the first ω moves encode a sequence $\sigma\in C$ then player W may win within 9 moves (no matter whether I or II is acting as W) and if they encode a sequence $\sigma\notin C$ then player B may win. This property ensures the non-determinacy of G, otherwise some player would be able to win the 0—1 game with C by a strategy.

We call the reader's attention to the problem of the ω-th move. It can be taken by either W or B and we may assume the worst, i.e., it is the turn of the one who is going to lose.

Summarizing, let $\langle B, B^0, B^1\rangle$ be an ω-brush, $(s)_W$ and $(s)_B$ for $s\in{}^{\underset{\smile}{\omega}}2$ be the subsets of B as defined in 4.1. For every $\sigma\in C$ we take two instances of a game similar to the second example in Proposition 2.5 with boards A^0_σ and A^1_σ. Let the elements of A^l_σ be R and P^i_j $(i\in\omega,\ j\leqq 7)$ and let the elements of the family $F^{0,l}_\sigma$ be $\{R, P^i_0, P^i_1\}\cup(\sigma|i)_W$, $\{R, P^i_0, P^i_2, P^i_3\}\cup(\sigma|i)_W$, $\{R, P^i_0, P^i_2, P^i_4, P^i_5\}\cup(\sigma|i)_W$ and $\{R, P^i_0, P^i_2, P^i_4, P^i_6\}\cup(\sigma|i)_W$ for all $i\in\omega$. (The branches of the i-th tree over the sequence σ are validated by the i-th cut of σ.) Let moreover the elements of the family $F^{1,l}_\sigma$ be $\{P^i_1, P^i_3, P^i_5, P^j_7\}$ for all $i\leqq j\in\omega$. Let $A=\bigcup\{A^l_\sigma\colon \sigma\in C,\ l=0, 1\}$.

The board of G is $A\cup B$, the family of W-winning sets is

$$F^0 = B^0\cup\bigcup\{F^{0,l}_\sigma\colon \sigma\in C,\ l=0,1\}$$

and the family of B-winning sets is

$$F^1 = B^1\cup\bigcup\{F^{1,l}_\sigma\colon \sigma\in C,\ l=0,1\}\cup\{\text{six-element subsets of } A\}.$$

We leave to the reader to check that the $\mathfrak{B}$-game $[A\cup B, F^0, F^1]^{\omega+20}$ has all the described properties. ∎

This construction cannot give a nondetermined $\mathfrak{A}$-game with countable board because every 0—1 game with countable winning set is determined. The following theorems deal with the case of countable boards.

THEOREM 4.3. *If there is a Ramsey cardinal then every $\mathfrak{A}$-game $[\omega, F]^{\omega\cdot 2}$ is determined.*

PROOF. The existence of a Ramsey cardinal implies that every Σ^1_1 game is determined [7]. It is easy to check that the family of ω-long plays after which I can win within finitely many moves forms a Σ^1_1 set. By the assumption this set is determined, i.e., either I has a strategy to remain in it, in which case I has a WS, or II has a strategy to avoid it, which means a DS for II. ∎

The axiom of constructibility $V=L$ implies the existence of a non-determined Σ_1^1 game [9]. The existence of a non-determined Σ_1^1 game, however, seems not to imply the existence of a non-determined $[\omega, F]^{\omega\cdot 2}$ game. But the construction can be carried over.

We proceed with two lemmas.

LEMMA 4.4. *Let W and B play two instances G and $\bar{G}$ of an unsymmetric game* [2] *as follows. First W chooses a finite 0—1 sequence $s_0\in \underset{\smile}{\omega}2$ then B chooses a digit $t_0\in\{0,1\}$ and a finite 0—1 sequence $\bar{s}_0$. W responds by choosing a digit $\bar{t}_0\in\{0,1\}$ and a finite 0—1 sequence s_1, etc. Thus the players form two infinite 0—1 sequences, $\sigma=s_0{}^\frown t_0{}^\frown s_1{}^\frown t_1{}^\frown\ldots$ and $\bar{\sigma}=\bar{s}_0{}^\frown\bar{t}_0{}^\frown\bar{s}_1{}^\frown\bar{t}_1{}^\frown\ldots$ where $^\frown$ denotes concatenation.*

Let $C\subset{}^\omega 2$ be uncountable and containing no perfect subset. If W plays by a strategy, then B has a counterplay such that $\sigma\notin C$ and $\bar{\sigma}\in C$. If B plays by a strategy then W has a counterplay such that $\sigma\in C$ and $\bar{\sigma}\notin C$.

PROOF. Assume that W plays by a strategy S, the other case is similar. The moves of W in $\bar{G}$ depend not only on the moves of B in $\bar{G}$ but on the moves in G, too. In view of this, we define countable sets X_n, Y_n for $n\in\omega$ by induction which satisfy the following conditions. The elements of X_n are sequences of the form

$$\langle t_0, \bar{s}_0, \bar{t}_1, t_1, \bar{s}_1, \bar{t}_2, \ldots, t_{n-1}, \bar{s}_{n-1}, \bar{t}_n\rangle$$

where $t_i, \bar{t}_{i+1}\in\{0,1\}$ and $\bar{s}_i\in\underset{\smile}{\omega}2$ for $i<n$ such that $\bar{t}_{i+1}$ is the response of W by S to the sequence of moves $t_0, \bar{s}_0, t_1, \bar{s}_1, \ldots, t_i, \bar{s}_i$ of B. Y_n is a subset of ${}^\omega 2$ and for every $\tau\in{}^\omega 2-Y_n$ and for every $t_i\in\{0,1\}$ $(i<n)$ there is exactly one sequence

$$\langle t_0, \bar{s}_0, \bar{t}_1, \ldots, t_{n-1}, \bar{s}_{n-1}, \bar{t}_n\rangle\in X_n$$

with the given digits t_i such that $\bar{s}_0{}^\frown\bar{t}_1{}^\frown\ldots{}^\frown\bar{s}_{n-1}{}^\frown\bar{t}_n\prec\tau$.

Let $X_0=\{\langle\rangle\}$, $Y_0=\emptyset$ and suppose that X_n, Y_n are defined and have the described properties. Then for every

$$s=\langle t_0, \bar{s}_0, \bar{t}_1, \ldots, t_{n-1}, \bar{s}_{n-1}, \bar{t}_n\rangle\in X_n$$

and $t_n\in\{0,1\}$ let X_{n+1} contain the sequences

$$\begin{gathered} s^\frown t_n^\frown\langle\rangle^\frown\bar{t}^0_{n+1}\\ s^\frown t_n^\frown\langle 1-\bar{t}^0_{n+1}\rangle^\frown\bar{t}^1_{n+1}\\ s^\frown t_n^\frown\langle 1-\bar{t}^0_{n+1}, 1-\bar{t}^1_{n+1}\rangle^\frown\bar{t}^2_{n+1}\\ \vdots \end{gathered}$$

where $\bar{t}^0_{n+1}, \bar{t}^1_{n+1}, \ldots$ are the corresponding responses of W by the strategy S. Let Y_{n+1} contain the elements of Y_n and the infinite sequences

$$\bar{s}_0^\frown\bar{t}_1^\frown\ldots{}^\frown\bar{s}_{n-1}{}^\frown\bar{t}_n{}^\frown\langle 1-\bar{t}^0_{n+1}, 1-\bar{t}^1_{n+1}, \ldots\rangle$$

with the above notations for every $s\in X_n$, $t_n\in\{0,1\}$.

Obviously, X_{n+1} and Y_{n+1} are countable if X_n and Y_n were, and they have the described properties. Therefore $\bigcup\{Y_n\colon n\in\omega\}$ is countable, C is not, so there is a $\tau\in C$ for which $\tau\notin Y_n$ for every $n\in\omega$.

Because X_{n+1} is an end-extension of X_n, we can define a "strategy" $\bar{S}$ of B as follows. Given $t_i\in\{0,1\}$ for $i<n$, let $\bar{S}(t_0,\dots,t_{n-1})$ be the only sequence for which

$$\langle t_0, \bar{S}(t_0), \bar{t}_1, t_1, \bar{S}(t_0,t_1), \bar{t}_2, \dots, t_{n-1}, \bar{S}(t_0,\dots,t_{n-1}), \bar{t}_n\rangle\in X_n$$

and

$$\bar{S}(t_0)^\frown\bar{t}_1^\frown\dots{}^\frown\bar{S}(t_0,\dots,t_{n-1})^\frown\bar{t}_n\prec\tau.$$

Playing by this "strategy" $\bar{S}$ in $\bar{G}$, B forces $\bar{\sigma}\in C$ independently of his moves in G. What is more, B forces to form always the same sequence $\bar{\sigma}=\tau$.

Now we turn our attention to G. Every move t_i of B in G determines uniquely $\bar{S}$ in $\bar{G}$ (via $\bar{S}$) so it determines uniquely W's response s_{i+1} in G. This means that we may forget about $\bar{G}$ totally and the strategy S reduces to a strategy S' for W in G only. Now the outcomes of plays according to S' form a perfect subset of ${}^{\omega}2$. C contains no perfect subset so B can choose a counterplay which lead out of C. Combining this counterplay with $\bar{S}$ we are done. ∎

LEMMA 4.5. *Assume $V=L$. There are finite trees T_s^0 and T_s^1 (trees in the set-theoretical sense, see* [3]*) for every $s\in{}^{\underset{\smile}{\omega}}2$ such that* (i) *and* (ii) *below are satisfied.*

(i) *If $s,t\in{}^{\underset{\smile}{\omega}}2$ and $s\prec t$ then T_t^l is an end-extension of T_s^l.*

By (i), *$T_\sigma^l=\bigcup\{T_s^l\colon s\prec\sigma\}$ is a tree of height $\leqq\omega$ for every $\sigma\in{}^{\omega}2$.*

(ii) *Let W and B play an infinite* 0—1 *game and let $\sigma\in{}^{\omega}2$ denote the resulting sequence. If W plays by a strategy then B has a counterplay such that either W picks only* 1 *and B picks only* 0 *after finitely many moves, or T_σ^0 is well-founded and T_σ^1 is not. If B plays by a strategy then W has a counterplay such that either B picks only* 1 *and W picks only* 0 *after finitely many moves, or T_σ^1 is well-founded and T_σ^0 is not.*

PROOF. $V=L$ implies that there is a Π_1^1 subset $C\subset{}^{\omega}2$ of cardinality 2^ω without a perfect subset [9]. Given any Π_1^1 subset $C\subset{}^{\omega}2$ one can assign finite trees T_s to every $s\in{}^{\underset{\smile}{\omega}}2$ such that $\bigcup\{T_s\colon s\prec\sigma\}$ is well-founded if and only if $\sigma\in C$, see [10]. By these facts and by the previous lemma, we are done if we can code the twofold unsymmetric game G of Lemma 4.4 in a single 0—1 game G^*.

But this latter task is easy. Enumerate all finite 0—1 sequences. Say W (or B) wants to choose in G some finite sequence s. Suppose s is the n-th in the enumeration then W (or B) chooses n consecutive 1's followed by a 0 in G^*. The single moves in G correspond to single moves in G^*. Clearly, every position of G^* determines uniquely the status of the simulation and every play in G^* corresponds to some play in G except for those where W or B picks only 1 after finitely many turns. ∎

THEOREM 4.6. *Assume $V=L$. There is a non-determined $\mathfrak{A}$-game $[\omega, F]^{\omega\cdot 2}$.*

PROOF. As in the proof of Theorem 4.2, we shall make a $\mathfrak{B}$-game G with these properties. We start with an ω-brush $\langle B, B^0, B^1\rangle$ and let $(s)_W$ and $(s)_B$ for $s\in{}^{\underset{\smile}{\omega}}2$ be the subsets of B as defined in 4.1. We define the remaining part of G with the help of the previous lemma. There are two essentially different cases.

Case A. The players in the first ω moves encode an exceptional sequence τ. For $k=0,1$ let $C^k\subset{}^{\omega}2$ be defined by

$$\langle d_0, d_1, \dots\rangle\in C^k \quad\text{iff}\quad d_{2i}=k,\quad d_{2i+1}=1-k \quad\text{for every}\quad i\geqq i_0.$$

Obviously, $\tau\in C^0\cup C^1$, $C^0\cup C^1$ is countable, and $\tau\in C^0$ if B plays by a strategy, $\tau\in C^1$ if W plays by a strategy. Let A_σ^l, $F_\sigma^{0,l}$, $F_\sigma^{1,l}$ $(l=0, 1)$ be as in the proof of 4.2, let

$$A_E^k = \bigcup\{A_\sigma^l\colon\ \sigma\in C^k,\ l=0,1\}\quad (k=0,1)$$

be the boards for the exceptional cases, and let

$$E^k = \bigcup\{F_\sigma^{1-k,l}\colon\ \sigma\in C^k,\ l=0,1\}\cup\bigcup\{F_\sigma^{k,l}\colon\ \sigma\in C^{1-k},\ l=0,1\}\cup$$
$$\cup\{21\text{-element subsets of } A_E^{1-k}\}$$

be the W winning sets for $k=0$, and the B winning sets for $k=1$. The boards are countable, and this part of the game has the following properties.

Suppose W plays by a strategy (the other case is quite similar). If they encode a sequence $\tau\in C^1$ then B can win at his fifth move in A_E^1. Moreover, B has a three-step threat in A_E^1 so either W loses or W has to threaten to win within three steps. (Remember, the ω-th step is W's turn.) Well, W cannot threaten in A_E^1 and cannot threaten in the remaining parts of the game. So W must play in A_E^0. But B can fend off every threat in A_E^0 and after 21 pairs of moves B wins eventually.

If the encoded sequence $\tau\notin C^0\cup C^1$ (observe, $\tau\in C^0$ cannot occur if B plays properly) then W's moves in A_E^0, as before, do not count and there can be at most 21 of them. The W's moves in A_E^1, however, cause a little problem. Here W wins after his 21-st move but he gives B ten free moves.

Indeed, W can threaten at his every other move only, and B can fend these threats off by one move. So B must be able to win at the other parts of the game with 10 free, but not necessarily consecutive moves.

Case B. Otherwise. Let T_s^l be the trees for $s\in{}^{\underset{\smile}{\omega}}2$, $l=0, 1$ the existence of which was shown in Lemma 4.5. Let T^0 and T^1 be disjoint trees of height ω such that every node has countably many immediate successors. If v is a node in T^l then $p(v)$ denotes its immediate predecessor and $h(v)\in\omega$ denotes its height: if v is the root then $h(v)=0$, otherwise $h(v)=h(p(v))+1$.

We may assume that T_s^l are embedded in T^l so that T^l is an end-extension of T_s^l and $T_s^l\cap T_t^l=T_r$ where r is the longest common initial segment of s and t. Therefore for each node v of T^l we can define a sequence $s_v^l\in{}^{\underset{\smile}{\omega}}2$ such that $v\in T_s^l$ if and only if $s_v^l=s$ or $s_v^l\prec s$.

Replace each node v in T^l by a broom associated with the set of immediate successors of v. These brooms will be fitted together in such a way that the players must climb on branches of the trees (Fig. 7). First B has a threat in T^0 and W can fend this off only by occupying a "validated" edge starting from the root of T^0. Doing so W has a threat in T^1 and B fends it off occupying a "validated" edge of T^1, etc. The validatings are done at the first ω steps. If they encode in the ω-brush the sequence $\sigma\in{}^\omega 2$ then the valid edges are just that of the subtrees $T_\sigma^l=\bigcup\{T_s^l\colon\ s\prec\sigma\}$. Now it is clear that if T_σ^0 is well-founded and T_σ^1 is not then W cannot fend all the threats of B off because W "runs out" of his tree eventually. It means that B wins within finitely many moves. Similarly, if T_σ^1 is well-founded and T_σ^0 is not, then W can win within finitely many moves. The exact definition of this part goes as follows.

Let T^l be the trees and s_v^l the sequences as discussed above. Let $\mathscr{B}_{l,v}=\langle B_{l,v}, B_{l,v}^0, B_{l,v}^1, C_{l,v}^0, C_{l,v}^1, D_{l,v}^0, D_{l,v}^1\rangle$ be the broom associated with the immediate

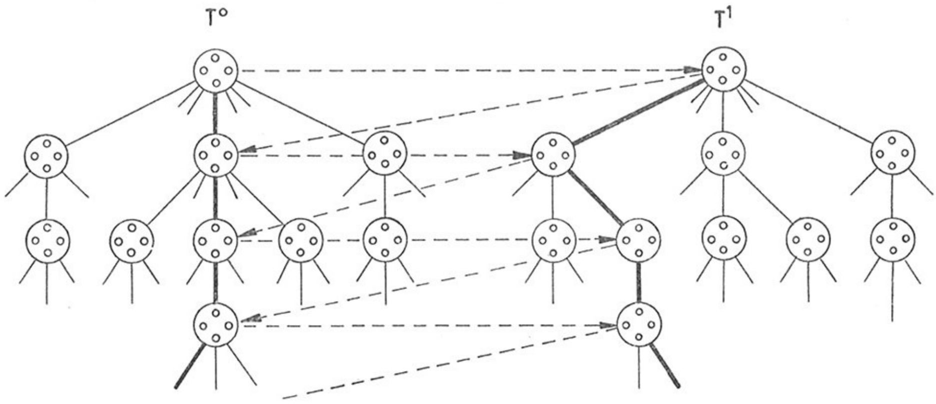

Fig. 7

successors of the node v of T^l. Let

$$A_M = \cup\{B_{l,v}\colon\ v \text{ is a node of } T^l,\ l = 0, 1\}$$

the board for the main case. We recall that for any node v, $p(v)$ denotes its only immediate predecessor and $h(v)$ denotes its height. Let, by definition, $C^k_{l,p(v)}(v)=\emptyset$ if v is the root of T (i.e., if $p(v)$ does not exist) and let

$$H^k_{l,v} = \begin{cases} \{\emptyset\} \text{ if } l=1 \text{ and } v \text{ is the root of } T^0 \\ \cup\{D^k_{l,w}\colon\ w \text{ is a node of } T^l \text{ and } h(w)=h(v)-l\} \text{ otherwise.} \end{cases}$$

We define the families $M^k_{l,v}$ for every node v of T^l and $k, l=0, 1$ as follows.

$$M^0_{0,v} = \{X\cup Y\cup C^0_{0,p(v)}(v)\cup (s^0_v)_W\colon\ X\in B^0_{0,v},\ Y\in H^1_{1,v}\}$$

$$M^0_{1,v} = \{X\cup Y\cup C^1_{1,p(v)}(v)\colon\ X\in B^1_{1,v},\ Y\in H^0_{0,v}\}$$

$$M^1_{0,v} = \{X\cup Y\cup C^1_{0,p(v)}(v)\colon\ X\in B^1_{0,v},\ Y\in H^0_{1,v}\}$$

$$M^1_{1,v} = \{X\cup Y\cup C^0_{1,p(v)}(v)\cup (s^1_v)_B\colon\ X\in B^0_{1,v},\ Y\in H^1_{0,v}\}.$$

Finally, let

$$M^k = \cup\{M^k_{l,v}\colon\ v \text{ is a node of } T^l,\ l = 0, 1\}$$

be the W winning sets for $k=0$ and the B winning sets for $k=1$.

The main part $\langle A_M, M^0, M^1\rangle$ has the following properties. The board, A_M is countable. We assume that always W picks the first element of A_M. At any moment, both W and B have one or two-step threats so any free move means victory immediately or at the next move. If W plays by strategy then B may win within finitely many moves, and if B plays by a strategy then W may win within finitely many moves (but only if W starts picking the elements of A_M).

This main part cannot be put over an ω-brush because there are two-step threats in it. By Case A, it must not contain even a four-step threat. And, what is the worst, it cannot be assumed that W starts picking the points of A_M. To solve these problems, we double the main part and put a little "prelude" before them as it was done in Lemma 3.8. Doing so we increase the size of the winning sets and assure that W starts picking.

Let $\langle A_M, M^0, M^1\rangle$ and $\langle A_N, N^0, N^1\rangle$ be two disjoint instances of the main part, U_M and U_N be disjoint 8-element sets. The family of i-element subsets of U is denoted by $[U]^i$. Let

$$F = A_M \cup A_N \cup U_M \cup U_N$$

be the board, and let

$$F^0 = [U_N]^5 \cup [U_M]^5 \cup \{X \cup Y:\ X \in M^0 \text{ and } Y \in [U_M]^4 \text{ or } X \in N^0 \text{ and } Y \in [U_N]^4\}$$

$$F^1 = \{X \cup Y:\ X \in [U_M]^4,\ Y \in [U_N]^4\} \cup$$

$$\cup \{X \cup Y:\ X \in M^1 \text{ and } Y \in [U_M]^4 \text{ or } X \in N^1 \text{ and } Y \in [U_N]^4\}$$

be the W winning and B winning sets. Every winning set in $F^0 \cup F^1$ has at least five elements. Now suppose W plays by a strategy, i.e., B is going to win. If W picks an element of $A_M \cup U_M$ ($A_N \cup U_N$) then B picks simply an unoccupied element of U_M (U_N). Then B either wins or validates just one of the parts A_M or A_N. Clearly, 8 free moves mean a victory for B during this prelude and 4 free moves do the same during the continuation.

At last if B plays by a strategy then the ω-th move belongs to B. If the first move of B in F is an element of $A_M \cup U_M$ ($A_N \cup U_N$) then W starts picking elements of U_N (U_M). W either can pick 5 elements of this (which means a victory because B has no 5-element winning set here), or B picks elements of U_N only, i.e., W can start playing in A_N.

Summarizing, the $\mathfrak{B}$-game $[B \cup A_E^0 \cup A_E^1 \cup F,\ B^0 \cup E^0 \cup F^0,\ B^1 \cup E^1 \cup F^1]^{\omega\cdot 2}$ is a non-determined $\mathfrak{B}$-game the existence of which was stated. ∎

In the last section of our paper we deal with the possible lengths of $\mathfrak{A}$-games in which I has WS.

PROPOSITION 4.7. *Let $\alpha = \omega\cdot\beta + n$ ($n \in \omega$) be an ordinal. Suppose there is an $\mathfrak{A}$-game $[A, F]^\alpha$ in which* I *has WS but has no α-bounded WS. Then the following cases cannot occur:*
(i) *$n > 0$ and n is even;*
(ii) *$\beta \neq 0$ and $n = 1$;*
(iii) *β is a limit ordinal and $n = 3$.*

PROOF. I cannot win after II's move so (i) follows. If I wins by occupying an element at limit step then I can occupy that element before, which proves (ii). Now, suppose I wins by his second move after a limit step. He can do it only if he has a "valid" V as shown in Fig. 8. This V became valid previously. Since β is a limit ordinal, I has a limit step before the $(\omega\cdot\beta)$-th when this V is valid, here I can occupy the bottom of the V and win. ∎

Fig. 8

THEOREM 4.8. *For every ordinal α not excluded in 4.7 there is an $\mathfrak{A}$-game $[A, F]^\alpha$ such that* I *has WS but has no α-bounded WS.*

PROOF. Let ξ be any ordinal and let $\xi=\omega\cdot\beta+n$ $(n\in\omega)$. ξ is even (odd) if n is even (odd). We define the ξ-train $\langle T, T^0, T^1\rangle$ for every odd ordinal ξ. The elements of T are the points P_δ for $\delta<\xi$. T^0 contains the subsets $\{P_\delta, P_{\delta+1}\}$ where $\delta+1<\xi$, δ is even, and if $n\geqq 5$ then the subset $\{P_{\omega\cdot\beta+k}: k<n,\ k \text{ is even}\}$ and if $n=3$ and $\beta=\gamma+1$ then the subset $\{P_{\omega\cdot\gamma}, P_{\omega\cdot\beta}, P_{\omega\cdot\beta+2}\}$. Finally, T^1 contains the subsets $\{P_\gamma, P_\delta\}$ where $\delta\leqq\gamma+1<\xi$, δ is even and γ is odd. The train is led by the 6-point engine E. The points of E are U_i, V_i for $i\leqq 2$, see Fig. 9. Let $U=\{U_0, U_1, U_2\}$ and $V=\{V_0, V_1, V_2\}$.

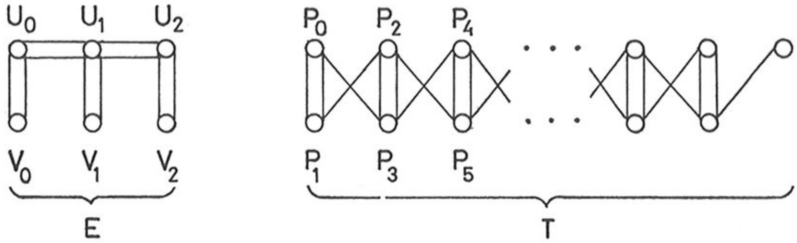

Fig. 9

Now let α be given and β, n be defined by $\alpha=\omega\cdot\beta+n$. We give the $\mathfrak{A}$-games which satisfy the requirements. The easy work of verification is left to the reader. We distinguish three cases.

Case A: α is finite. This case is trivial.

Case B: $\alpha=\omega\cdot\beta+n$ is infinite, n is odd and either $n>3$ or $n=3$ and $\beta=\gamma+1$.

Let E be the engine, $\langle T, T^0, T^1\rangle$ be the α-train. The board of the $\mathfrak{A}$-game is $E\cup T$ and the winning sets are $\{U_0, V_0\}$, $\{U_0, U_1, V_1\}$, $\{U_0, U_1, U_2, V_2\}$, $U\cup X$ for all $X\in T^0$ and $V\cup Y$ for all $Y\in T^1$. I may win picking $U_0, U_1, U_2, P_0, P_2, \dots$ in succession. If I does not do so, II can either win or make a draw.

Case C: α is a limit. Let $\alpha=\sup\{\alpha_i: i\in I\}$, $\alpha_i=\omega\cdot\beta_i+n_i$ such that n_i is odd and $n_i>3$. Let $\langle T_i, T_i^0, T_i^1\rangle$ be the α_i-train, E be the engine and $\langle B, B^0, B^1, C^0, C^1, D^0, D^1\rangle$ be the broom associated with the index set I, and R be an entirely new point. The board of the $\mathfrak{A}$-game is $E\cup B\cup\{R\}\cup\bigcup\{T_i: i\in I\}$. The winning sets are $\{U_0, V_0\}$, $\{U_0, U_1, V_1\}$, $\{U_0, U_1, U_2, V_2\}$, $V\cup\{R\}$, the sets $U\cup X$ and $V\cup Y$ for all $X\in B^1$ and $Y\in B^0$, and the sets

$$U\cup C^1(i)\cup X\cup Y \quad \text{for} \quad X\in D^1,\ Y\in T_i^0$$

$$V\cup C^0(i)\cup X\cup Y \quad \text{for} \quad X\in D^0,\ Y\in T_i^1$$

for all $i\in I$.

After playing off the engine, I has to pick the point R, and II may choose any train to play in. Therefore there are plays of length at least α_i, i.e., I has no α-bounded WS. On the other side I can obviously win before the α-th step. ∎

The cardinality of the boards in these constructions is equal to the cardinality of α if α is infinite. This is the smallest possible value if α is not a successor cardinal. If $\alpha=\varkappa^+$ where $\varkappa$ is an infinite cardinal then the lower bound is $\varkappa$. We can construct $\mathfrak{A}$-games $[\varkappa, F]^{\varkappa^+}$ where I has WS but has no $\varkappa^+$-bounded WS, but the construction is too difficult to give here.

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