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Variations on a Game

J. BECK AND L. CSIRMAZ

Mathematical Institute of the Hungarian Academy of Sciences, Budapest, Reáltanoda u 13-15 Hungary – 1053

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0. INTRODUCTION

It is beyond doubt that we are witnessing a renaissance of game theory. Besides the traditional theory, founded by J. von Neumann and O. Morgenstern, it suffices to mention the advances in the theory of NIM-type games, more precisely, in the addition theory of partizan games; see J. H. Conway, All games bright and beautiful, *Amer. Math. Monthly* (1977) 417–434. Here, however, we shall outline some results from a quite different, but no less interesting branch of game theory.

To see where our concepts arose, inspect two well-known particular games. The first is a version of the Oriental Go-Moku, the game 5-in-a-row. It was the subject of the 10th problem dedicated to the memory of Starke, *Amer. Math. Monthly* (1979) p. 306; the solution appeared in (1980) 575–576. The game is played on an infinite chessboard. Two players, whom we shall call Maker and Breaker (the names will be justified later) move alternately with Maker going first. Each of them in turn occupies one previously unoccupied square of the chessboard. The winner is the first one to occupy 5 or more adjacent squares in a row, horizontally, vertically, or diagonally.

An easy argument shows the Breaker cannot have a winning strategy, and either Maker has a winning strategy, or else Breaker has a drawing strategy, i.e., a strategy which allows Breaker to play indefinitely. It is a folklore that Maker wins this game but the authors know of any strict proof. It is not even known whether a winning strategy for Maker, if any, can be bounded in time. In other words, assume that Maker has a winning strategy. Is there then a natural number n such that Maker can win before his nth move? The best result of this kind is due to Keisler [16]. If the game is played on countably many boards, i.e., at every step the player chooses one of the boards and occupies a square on it, and if Maker has a winning strategy, then Maker has a winning strategy bounded in time.

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A very similar game is tic-tac-toe alias noughts-and-crosses. "Every child knows how to play this game" writes Dudeney in his famous book [9]. "You make a square of nine cells, and each of the players, playing alternately, puts his mark (a nought or a cross, as the case may be) in a cell with the object of getting three in a line. Whichever player first gets three in a line, wins with the exulting cry:

Tit, tat, toe,
My last go;
Three jolly butcher boys
All in a row."

Several generalizations of this game are studied in the literature [4, 8, 15, 20]; we shall also discuss it later.

The other game we shall considere here is Shannon's switching game. It has many variants and we have picked the one which seemed to best fit our purposes. The game is played on a connected graph G. The players, Maker and Breaker, alternately occupy preciously unoccupied edges of G with Maker going first. Maker's aim is to pick all the edges of some cut of G, i.e., edges whose deletion splits G into more components. Breaker's aim is simply to prevent Maker from achieving his goal.

Lehman observed [17] that the existence of two edge-disjoint spanning trees of G is equivalent to the existence of a winning strategy for Breaker. Tutte [21] and Nash-Williams [19] proved independently that if there are no such spanning trees, then for some subset A of the edges, the deletion of A splits G into at least $(|A|+3)/2$ components. When such a subset exists, Maker wins simply by occupying edges from A as long as there remain any.

Though these games are different in many respects, they are closely related. In both games a hypergraph is given, i.e., a collection of sets. These sets are the *edges*, and the elements of the edges are the *vertices* of the hypergraph. The players alternately occupy previously unoccupied vertices. In both games Maker makes the first move, and Maker's aim is to occupy all the vertices of some edge. Breaker's aim, however, is different. In the game 5-in-a-row he wants to pick all the points of some edge *before* Maker can do it, whilst in the switching game he doesn't want Maker to achieve his goal.

The games whose general pattern have been outlined here belong to the class of positional games, and where studied by Hales and Jewett [15], Erdös and Selfridge [13], Berge [6], and others, We call them amoeba games after the Hungarian name of 5-in-a-row, and since almost all games mentioned in this paper are amoeba games, this attributive will often be omitted.

We distinguish strong and weak (amoeba) games according to Breaker's aim as follows: In the *weak* version Maker's aim is to pick every vertex of some edge of a given hypergraph H, and Breaker's aim is to prevent him from doing so. The winner is the one who achieves his goal. In the *strong* version both players want to pick every vertex of some edge of H, the winner is the one who does it sooner. If neither of them succeeds, and in any other case, the game is a draw.

Of course we have to speak about the length of the game. If there are only finitely many vertices in H, then there is no problem. The game ends if either players wins, or every vertex has been chosen. If H contains infinitely many vertices, then the players take turns until either of them wins, or until they have taken their nth turns for every natural number n. This agreement will be violated in the third part of this paper, the players will be allowed—and required—to continue their moves after these infinitely many moves.

We are interested in the existence of winning and drawing strategies, so we give a rough definition of these notions. Suppose the game is played on the hypergraph H. A *strategy* for Maker is a function S with domain the set of finite sequences of vertices of H (including the empty equence), such that $S(\langle w_0, w_1, \ldots, w_{n-1}\rangle)$ is always a vertex of H different from the vertices $v_i = S(\langle w_0, w_1, \ldots, w_{i-1}\rangle)$ and w_i for $i = 0, 1, \ldots, n-1$ (if there is any such vertex). In a play according to this strategy, Maker determines all his moves by S as follows: Suppose they have picked the vertices v_0, w_0, $v_1, w_1, \ldots, v_{n-1}, w_{n-1}$ in this order, then Maker's nth move is $v_n = S(\langle w_0, w_1, \ldots, w_{n-1}\rangle)$. The notions of a strategy for Breaker and a play according to a strategy for Breaker are similarly defined. The strategy S is a winning strategy for Maker (for Breaker) if every play according to S is a win for Maker (for Breaker). Strategy S is a drawing strategy, if every play according to S is either a win or a draw. A game is *determined* if either player has a winning strategy, or both of them has a drawing strategy.

The rest of this paper is divided into three parts. In the first part we shall discuss the weak amoeba games; the results are mainly of combinatorial character. The middle part deals with the strong games and with the connections, or rather with the differences between strong and weak games. In these parts we shall assume without further notice that the edges of the hypergraphs are finite. Finally, in the last part we shall give some possible generalizations towards the infinite.

1. Weak Amoeba Games

Throughout this section, log denotes the natural logarithm, $\exp_k$ denotes the k-fold iteration of the exponential function e^x, and the inverse of $\exp_k$ is denoted by $\log_k$.

As a warm-up, we shall announce some results for three further games (cf. [3, 4]).

Van der Waerden Game

The weak amoeba game $W(N, n)$ is played on the hypergraph whose edges are the arithmetic progressions of n terms from the interval $\{0, 1, \ldots, N-1\}$. In other words, in $W(N, n)$ Maker and Breaker alternately say natural numbers below N with the proviso that no number can be mentioned twice. Maker wins if he mentions (among others) all the elements of an A.P. of n terms.

THEOREM 1. *Let $\varepsilon > 0$ be arbitrary. For every N large enough depending only on ε, in the game* $W(N, n)$:

(i) *if* $n < (1-\varepsilon)(\log N/\log 2)$, *then Maker*;

(ii) *if* $n > (1+\varepsilon)(\log N/\log 2)$, *then Breaker has a winning strategy.* ∎

Ramsey Game

If S is a set, then $[S]^k$ denotes the family of subsets of S containing exactly k elements. Following the set-theoretical traditions, we identify the natural number N with the set of its predecessors, i.e., $N = \{0, 1, \ldots, N-1\}$. So $[N]^2$ can be regarded as a complete graph with N vertices. The players alternately occupy edges of this graph (i.e., elements of $[N]^2$) and Maker's aim is to pick all the edges of a complete subgraph with n vertices (i.e., all the elements of $[S]^2$ for some n-element subset of S of N). This weak game is denoted by $R(N, n)$.

The game $R_k(N, n)$ is a trivial generalization. The players alternately occupy k-element subsets of N, and Maker wins if he picked all the elements of $[S]^k$ for some n-element subset S of N. The following results are partially due to Erdös and Selfridge [13]:

THEOREM 2. *For every $k \geqq 2$ there are positive constants c_k and c_k' such that in the game* $R_k(N, n)$:

(i) *if* $n < c_k(\log N)^{1/k}$, *then Maker*;

(ii) *if* $n > c_k'(\log N)^{1/(k-1)}$, *then Breaker has a winning strategy.* ∎

In case of $k = 2$ the breaking point is known to be within the closer bounds $(1/\log 2 - \varepsilon)\log N < n < 2\log N/\log 2$.

Hales–Jewett Game

This is a straightforward generalization of the game tic-tac-toe. The players alternately put their marks in the cells of a d-dimensional cube of size $n \times n \times \cdots \times n$. Maker wins if he has n of his marks *in a line*. More precisely, the board of the game $\mathrm{HJ}(d, n)$ is the set of d-tuples

$$\begin{aligned} B &= n \times n \times \cdots \times n \\ &= \{\langle a_1, a_2, \ldots, a_d \rangle : 0 \leqq a_j < n \text{ for each } 1 \leqq j \leqq d\}. \end{aligned}$$

The edges of the hypergraph on which $\mathrm{HJ}(d, n)$ is played are those n-element subsets $\{\langle a_1^i, a_2^i, \ldots, a_d^i \rangle : 0 \leqq i < n\}$ of the board B such that, for each j, the sequence $\langle a_j^0, a_j^1, \ldots, a_j^{n-1} \rangle$ composed of the jth coordinates is either strictly increasing (from 0 to $n-1$), or strictly decreasing (from $n-1$ to 0), or constant.

It is easy to see that HJ(2, 3), which is the weak version of tic-tac-toe, is a win for Maker. The spatial HJ(3, 4) is also a win for Maker; this follows from the fact that Maker wins even the strong version, see Martin Gardner's column in Scientific American, 1980. The following theorem improves some results of Hales and Jewett [15], Erdös and Selfridge [13].

THEOREM 3. *Let $\varepsilon > 0$ be arbitrary. In the game* $\mathrm{HJ}(d, n)$

(i) *if $n^2(\log 2/2) < d$, then Maker*;

(ii) *if $n((\log 2/\log 3) - \varepsilon) > d$ and $n > n_0(\varepsilon)$, then Breaker has a winning strategy.* ▮

When Can Maker Win?

In these examples the sets of vertices of the hypergraphs corresponding to the games—the boards, in short— were finite. A game with finite board is determined since every play ends after finitely many moves. This determinateness remains valid if only the finiteness of the edges is assumed. Indeed, if Maker has no winning strategy, then Breaker can always make a move so that Maker still has no winning strategy. Since the edges are finite, if Maker wins he wins after finitely many moves, so Maker cannot win at all. The strategy described here is just a winning strategy for Breaker.

There is a rather general sufficient condition for Maker's win. To describe it we recall that the chromatic number of a hypergraph is the least integer r such that the vertices can be colored with r colors yielding no monochromatic edge.

PROPOSITION 4. *Suppose that the set of vertices of the hypergraph H is finite, and the chromatic number of H is at least* 3. *Then Maker has a winning strategy in the weak game played on H.* ▮

The latter condition is sometimes expressed by saying that H does not have *property B*.

Unfortunately, this result is rather weak for the majority of applications. Consider, e.g., the van der Waerden game. The finite form of van der Waerden's well-known theorem states that for every positive integer n there exists a smallest integer $f(n)$ with the following property: If the natural numbers less than $f(n)$ are arbitrarily colored with two colors, then there is a monochromatic arithmetic progression of n terms. From Proposition 4 it follows immediately that $N \geqq f(n)$ implies the existence of Maker's winning strategy in $\mathrm{W}(N, n)$. The best upper bound on $f(n)$ known at present, however, is extremely poor; e.g., it is an open problem whether $f(n) < \exp_k n$ holds for some constant k.

The situation is very similar in the case of Ramsey games. Let $g_k(N)$ denote the largest number n such that, for every 2-coloring of the elements of $[N]^k$, there exists an n-element subset S of N whose every k-element subset has the same color; i.e., there exists $S \in [N]^n$ such that $[S]^k$ is monochromatic. The value of $g_k(N)$ has a quite different order of magnitude than the breaking point for the game $\mathrm{R}_k(N, n)$. More exactly, for $k \geqq 3$, there are positive constants d_k and d_k' such that

$$d_k \log_{k-1} N < g_k(N) < d_k' \log_{k-2} N,$$

see [10, 12]. The following theorem, however, gives a condition for Maker's win which is already strong enough to impy some of the announced results [4].

THEOREM 5. *Let H be an n-uniform hypergraph (every edge has exactly n vertices), and let v be the number of the vertices in H. Suppose moreover that, fixing two vertices, no more than d edges contain both of them. If $|H| > vd2^n$, then Maker has a winning strategy in the weak game played on H.* ▮

Theorem 1(i) can be deduced easily. Indeed, there are more than $N^2/4n$ arithmetic progression of n terms in the interval $\{0, 1,..., N-1\}$, and at most $\binom{n}{2}$ such A.P. can contain two fixed integers. Therefore, if $N^2/4n > N\binom{n}{2} 2^n$, then Maker has a winning strategy, and this inequality evidently holds if $N > n^4 2^n$. Similar but more complicated computations give the other announced results for Maker's win.

The condition given in Proposition 4 is sufficient not only for finite but for arbitrary hypergraphs with finite edges. Indeed, suppose that the chromatic number of H is at least 3. An easy generalization of a theorem of de Bruijn and Erdös [7] gives that there are finitely many edges of H such that the subhypergraph G composed from them has chromatic number $\geqq$3. By Proposition 4, Maker has a winning strategy in G, and he can play—and win—by this strategy in H, too, he simply ignores those moves of Breaker which are not vertices of G.

In this construction Maker wins the game because he can win in some finite part. This compactness property is valid in general.

PROPOSITION 6. *The weak game played on H is a win for Maker if and only if for some finite subhypergraph G of H, the game played on G is a win for Maker.*

Proof. The *if* part is obvious. To show the *only if* part, assume that Breaker wins on every finite subhypergraph $G \subset H$, and fix a winning strategy S_G for Breaker in each G. We combine these S_G's to get a winning strategy for Breaker in the big game. To do this let A be the family of finite subhypergraphs of H, and choose an ultrafilter U on A such that for every $G \in A$ we have $\{G' \in A: G' \supset G\} \in U$. Now Breaker plays as follows: If there is some vertex v of H such that almost all (in the sense of U) of the strategies agree on v as the next move of Breaker, then he chooses v; if there is no such vertex, then he makes an arbitrary move. Now suppose that Maker wins the play in which the vertices $v_0, w_0, v_1, w_1, \ldots, v_{n-1}, w_{n-1}, v_n$ were chosen in this order, and Breaker picked all the vertices w_i by the strategy described above. Then there is some finite hypergraph $G \in A$ such that v_i and w_i are vertices in G, the set $\{v_0, v_1, \ldots, v_n\}$ contains an edge of G, and for every $0 \leqq i < n$, $S_G(\langle v_0, v_1, \ldots, v_i \rangle) \notin \{v_{i+1}, \ldots, v_n\}$. Thus picking the vertices $v_0, v_1, \ldots, v_n$ in this order, Maker wins against the strategy S_G, a contradiction. ∎

This proof of the theorem is due to Fred Galvin.

When Can Breaker Win?

We start with a fundamental combinatorial theorem due to Erdös and Selfridge [13].

THEOREM 7. *If H is an n-uniform hypergraph and $|H| < 2^{n-1}$, then Breaker has a winning strategy in the weak game played on H.*

Proof. Given a finite hypergraph G we assign the value

$$v(G) = \sum_{A \in G} 2^{-|A|}$$

to G. Consider a play on H in which the vertices $v_0, w_0, v_1, w_1, \ldots,$ were picked in this order. Define the hypergraphs H_i for $i \geqq 0$ as follows. Throw away those edges from H which contain any vertex picked by Breaker, and from the remaining edges throw away the vertices picked by Maker, i.e.,

$$H_i = \{A \setminus \{v_0, \ldots, v_i\}: A \in H \text{ and } A \cap \{w_0, \ldots, w_{i-1}\} = \varnothing\}.$$

Maker wins if and only if some of the H_i's contain the empty set, and since the cardinality of the empty set is zero, in this case $v(H_i) \geqq 1$. Thus if $v(H_i) < 1$ for every $i \geqq 0$, then Breaker wins.

We define a strategy for Breaker. Let the value of an edge $A \in H_i$ be $2^{-|A|}$, and the value of a vertex of H_i be the sum of the values of the edges it belongs to. In his ith move Breaker picks that vertex of H_i which is of largest value. We claim that $v(H_{i+1}) \leqq v(H_i)$ independently of Maker's (i + 1)st move. If we prove this the result follows since every edge of H_0 contains n or $n-1$ vertices, so $v(H_0) \leqq |H_0| \, 2^{-n+1} < 1$. Therefore, $v(H_i) < 1$ for every $i \geqq 0$.

We check $v(H_{i+1}) \leqq v(H_i)$. Before Maker's $(i+1)$st move, the sum of values of the edges is $v(H_i) - \varphi$, where φ is the value of the vertex w_i picked by Breaker. On Maker's next move he doubles the value of each edge containing his vertex v_{i+1}, so he adds to $v(H_i) - \varphi$ no more than the previous value ψ of v_{i+1}. But $\varphi \geqq \psi$ by the definition of w_i, so $v(H_{i+1}) \leqq v(H_i) - \varphi + \psi \leqq v(H_i)$ which was to be proved. ∎

Generalizations of this valuation method are at the heart of the other combinatorial results mentioned here. The theorem is sharp in the sense that for every n, an n-uniform hypergraph with 2^{n-1} edges can be constructed which is a win for Maker. For example, the full branches of a binary tree with n levels form such a hypergraph (Fig. 1).

From this theorem a somewhat weaker form of the second part of Theorem 1 follows immediately. There are fewer than N^2/n arithmetic progression of n terms from the interval $\{0, 1, ..., N-1\}$, so Breaker has a winning strategy in the game $\mathrm{W}(N, n)$ if $N^2/n \leqq 2^{n-1}$, i.e., if $n > (2+\varepsilon)(\log N/\log 2)$ for N large enough.

The hypergraph in the Hales–Jewett game $\mathrm{HJ}(d, n)$ is n-uniform and contains $\frac{1}{2}[(n+2)^d - n^d]$ edges. So Theorem 8 gives the rather weak sufficient condition $(n+2)^d - n^d < 2^n$ for Breaker's win. This hypergraph, however, has an important additional feature, namely, any two edges have at most one vertex in common. The hypergraphs with this property are called *almost disjoint*. For hypergraphs of this type the upper bound 2^{n-1} of Theorem 7 can be raised considerably [3].

FIGURE 1

THEOREM 8. *There is a constant $c > 0$ such that for every n-uniform almost-disjoint hypergraph H, if $|H| < 4^{n-c\sqrt{n}}$, then Breaker has a winning strategy in the weak game played on H.* ▮

This theorem is also sharp as far as the order of magnitude is concerned. Erdös and Lovász constructed [11] a 3-chromatic n-uniform almost-disjoint hypergraph with no more than $n^4 4^n$ edges. The game played on their hypergraph, as stated in Proposition 4, is a win for Maker.

On some n-uniform hypergraphs Maker can win very quickly, e.g., in the extremal game indicated in Fig. 1 Maker wins on his nth move. The situation is quite different with almost-disjoint hypergraphs; here maker cannot force a win within about 2^n moves.

THEOREM 9. *Let $\varepsilon > 0$ be arbitrary. If $n > n_0(\varepsilon)$ and H is an n-uniform almost-disjoint hypergraph, then Breaker has a strategy playing which he does not lose before his* $(2-\varepsilon)^n$th *move.* ▮

This result is also sharp in the asymptotic sense because the construction of Erdös and Lovász cited above has no more than $n^4 2^n$ vertices.

Biased Games

When a game is overwhelmingly in favor of one of the players, one can make up this handicap by allowing the other to pick many vertices in a move. Games so played are called *biased*. More precisely, let m and b be positive integers, and let H be a hypergraph. In the biased game (m, b, H) Maker and Breaker take moves alternately, with Maker going first. In each of his moves, Maker picks m vertices, and Breaker picks b vertices of H. The conditions for a win are the same as in amoeba games, and so we could distinguish weak and strong biased games, but we shall deal with weak games only. For these games a generalization of Theorems 5 and 7 holds as follows [5].

THEOREM 10. *Let H be an n-uniform hypergraph with v vertices such that fixing two vertices no more than d edges contain both of them. Then in the weak biased game (m, b, H):*

(i) *if $|H| > vdmb(1+(b/m))^n$, then Maker;*

(ii) *if $|H| < (1+b)^{(n/m)-1}$, then Breaker has a winning strategy.* ▮

The second part of this theorem is also sharp if m is a divisor of n. The frame of the extremal game is a tree of height n/m, in which every node has exactly $b+1$ immediate successors. Put m points in place of each node, and an edge of the extremal hypergraph is the union of points along a full branch. Obviously, it has $(1+b)^{(n/m)-1}$ edges and is a win for Maker.

From Theorem 10 one can get bounds easily for the breaking points of the biased versions of van der Waerden and Ramsey games, cf. [8]. Proposition 4, however, seems not to generalize immediately to this case, and we do not known whether every 4-chromatic hypergraph is a win for Maker in that biased game where Maker picks one vertex, and Breaker picks two vertices in each of his move[1].

2. Strong Amoeba Games

On the same hypergraph Maker and Breaker may play both weak and strong games. From the preceding section we know something about the odds in the weak version; let us see what happens in the strong games.

When a Draw is Worth a Win

With the restriction we have made at the beginning, i.e., that the edges are finite, the strong games are also determined; either one of the players has a winning strategy, or both of them can force a draw. An easy argument shows that Breaker cannot win against a clever Maker, so a strong game has only two possible outcomes: it is either a win for Maker, or is a draw. Breaker's only reasonable goal is not to lose, and in this respect the strong version resembles the weak one. If Breaker wins in the weak game, then the same play gives him (at least) a draw in the strong game. Therefore the results in the preceding section giving sufficient conditions for Breaker's win in the weak games yield conditions without any change for Breaker's draw in the strong case. If Breaker is satisfied with the draw, he has the advantage that in the strong game he can threaten, and Maker has to waste valuable moves fending off the threats. What is more, it is quite possible that Maker wins the weak version while Breaker can force a draw in the strong version. This happens, e.g., in the game tic-tac-toe: the original game is a draw but the weak version is a win for Maker.

While playing a strong game, both players have their own threats, and either of them, fending off the other's, may build his own. Therefore, a play is a delicate balancing between threats and counterthreats and can be of very intricate structure even if the hypergraph of the game is simple. Nevertheless, the claim of Proposition 4 remains valid.

Proposition 11. *Suppose that the set of vertices of the hypergraph H is finite, and the chromatic number of H is at least* 3. *Then Maker has a winning strategy in the strong game played on H.* ∎

This proof goes along the same lines as that of Proposition 4. As we have mentioned, this condition gives very poor estimates. For example, Hales and

[1] P. Frankel constructed a game of this type which is a win for Breaker. The problem in general, however, has remained open.

Jewett proved the existence of a threshold function $d_0(n)$ such that the hypergraph of the game HJ(d, n) has chromatic number at least 3 whenever $d > d_0(n)$ [15]. In this case Maker wins the strong version of the game HJ(d, n). On the other hand, Theorem 4 gives the condition $d < n((\log 2/\log 3) - \varepsilon) = d_1(n)$ for Breaker's draw. Unfortunately, no sensible estimate is known for the threshold function $d_0(n)$, and this is so because van der Waerden's cited theorem is an easy corollary of the theorem of Hales and Jewett. There is a wide gap between $d_1(n)$ and $d_0(n)$ and the behavior of the strong game is unknown in this interval. Even the existence of a breaking point is questionable. It may well happen that Maker wins the strong version of HJ(d, n), but the same game with $d+1$ instead of d is a draw.

The situation is very similar to the strong versions of van der Waerden and Ramsey games, and no powerful method is known for handling these problems.

Incompactness Properties

While Proposition 7 gave a nice compactness-type property for weak games, there is hardly any for the strong games. To start with, let G be a subhypergraph of H. It may happen that Maker wins the strong game on G while the game on H is a draw. This is the case if the edges of G are the 3-element subsets of some 5-element set, and H contains infinitely many disjoint two-element edges besides G. Notice that this H is 3-chromatic (because G is); therefore, Proposition 11 does not generalize for infinite hypergraphs.

The hypergraph sketched in Fig. 2 gives an example of a strong game in which Maker wins but Breaker has arbitrary long counterplay. The edges are the full branches of the trees T_n for $n \geqq 0$; here T_0 has eight 4-element branches, and T_n has exactly one i-element branch for every $3 \leqq i \leqq n+2$. Maker can pick elements of a branch of T_0, but Breaker may postpone his defeat for n moves by threatening in T_n.

In this example after Breaker's first move the length of the game is no longer in doubt. If Breaker picked a point from T_n, then Maker can win within $n+3$ turns but no sooner. Let us say that the *rank* of a game is n if

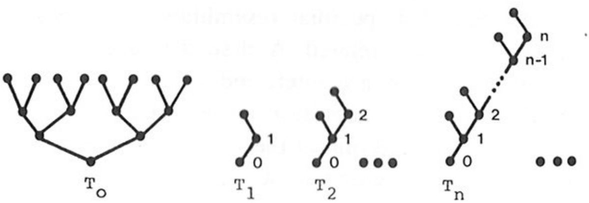

FIGURE 2

Maker can win within $n+1$ turns but no sooner. For example, if Maker wins with his first move, then the rank is 0. The game of Fig. 2 is not of rank n, but after the first turn it becomes a game of rank n for some natural number n. If we want to assign a rank to this game, too, then the best choice is the least ordinal exceeding every natural number, that is ω. In general, the rank of a game is an ordinal number and can be defined as follows: Say that a position of the game is *nice*, if the next move belongs to Maker (in other words, the players occupied even number of points), no player has won and Maker still has a winning strategy. We assign ordinal ranks to these nice positions. Let this number be 0 if Maker can win in one move; otherwise, let it be the least ordinal number α for which Maker has such a move that for Breaker's any move the new position (with two more occupied points) is nice and has rank less than α. The rank of a game is, of course, the rank of the starting position. It measures, in some sense, how far Maker is from victory.

Every amoeba game, which is a win for Maker, has a uniquely determined rank, and it is so because the edges of its hypergraph are finite. By Proposition 6 the ranks of weak games are always finite; the strong amoeba games, however, are of quite different character [1].

THEOREM 12. *For every ordinal number, there exists a strong game whose rank exceeds that ordinal.* ∎

Last but not least there is an increasing sequence $H_0 \subset H_1 \subset H_2 \subset \cdots$ of finite hypergraphs such that Breaker can force a draw in each strong game played on H_k, but Maker wins on their union $H = \bigcup_{k=0}^{\infty} H_k$ within five turns [1].

Snub Games

Every two-person game can be played in a *snub* way. Our players are, as usual, Maker and Breaker, and suppose that a given two-person game is played by White and Black. Maker and Breaker agree on playing this game, but before starting, as a *premove*, Maker chooses who he wants to be, White or Black. After this choice, they start playing the game according to its rules. Snub games are in favour of Maker, because here Breaker clearly cannot have a winning strategy. This peculiar resemblance to strong amoeba games suggests that they are closely related. And so they are.

The two-person games we are interested in are the so-called positional games. One of their possible definitions is as follows: A tree of height at most ω is given. It is supposed that at the lowest level there is only one node, the root. Every node may have immediate successors of arbitrary cardinality, and the nodes having no successor are labelled by one of the letters W or B. The game is played by White and Black with White going first. Initially a marker is placed into the root of the tree, and the players alternately push the marker one level higher along an edge. The game ends if the marker arrives (within finitely many moves) into a node with no successor, the winner is the one whose initial is its label. In the other cases the game is a draw. One may think of these nodes as the positions in some game (with a perfect description of the previous moves), and the edges of the tree indicate the legal moves. These games are determined and belong to the class of infinite open games of perfect information studied first by Gale and Stewart [14]. A snub positional game is a positional game played in a snub way.

THEOREM 13. *There is a uniform construction which gives for every snub positional game* G_0 *a strong amoeba game* G_1 *such that*:

(i) *the hypergraph of* G_1 *is effectively constructed from the tree of* G_0;

(ii) *from every winning (drawing) strategy for Maker (Breaker) in* G_i, *a winning (drawing) strategy can be effectively constructed for the same player in* G_{1-i} $(i=0, 1)$. ▮

If the tree of G_0 is defined by some finite description, then a finite description exists for the hypergraph of G_1. This is valid for the strategies, too, and leads to a surprising consequence. There exists a strong amoeba game in which Maker has a winning strategy, but if a computer, not matter how large and how fast, plays instead of Maker, then Breaker can win against it [1].

COROLLARY 14. *There is a strong amoeba game such that*:

(i) *the vertices of the hypergraph are the natural numbers*; *every edge contains fewer than* 100 *vertices*; *the edges form a decidable set, i.e., there exists a recursive procedure which decides whether a given finite set of natural numbers is an edge or not*;

(ii) *every play ends before the* 100th *turn, no matter how the players play*;

(iii) *Maker has a winning strategy* (*this follows, e.g., from* (ii)) *but Breaker may win against any recursive strategy of Maker.*

Proof. We give here only the idea behind the construction. It suffices to find a positional game with similar properties because the previous theorem allows us to salvage them. This latter task is easy. Let White and Black play as follows: First, White chooses a Turing machine M; then Black chooses a natural number x; finally White chooses a natural number y different from x. White wins if the machine M, applied to the blank tape, halts after exactly y steps; otherwise, Black wins. Clearly, Black has a winning strategy, but no recursive winning strategy. ▮

3. Beyond the Finite

Following the set-theoretical conventions introduced by J. von Neumann, we identify an ordinal number with the set of its predecessors, and a cardinal number with the smallest ordinal which, as a set, has that cardinality. So ω and $\aleph_0$ are the same ordinal, and if κ is a cardinal number, then κ itself is a set of cardinality κ. κ^+ is the smallest cardinal exceeding κ, and 2^κ is the cardinality of the power set (i.e., the set of all subsets) of κ. The generalized continuum hypothesis (GCH) is the assertion that $2^\kappa = \kappa^+$ holds for every infinite cardinal κ. It is well known that GCH is neither provable nor disprovable in ZFC, the usual axiom system of set theory.

More Moves

Let H be an arbitrary hypergraph, and suppose Maker and Breaker play an amoeba game on H, and they play not only until they have made their nth move for every natural number n, but they continue to make moves as long as there is any unoccupied vertex of H. In this case a play is a transfinite sequence of moves, and for an ordinal number α, the αth move of that play is the αth element of the sequence. The players move alternately, but the limit moves have no immediate predecessor so one has to decide separately about them. We admit the most natural possibility and offer these limit moves to Maker. This type of games will be called *infinite*.

For an ordinal number α, a game of length α is an infinite game with the additional rule that if neither player has won before the αth move, then the game is a draw.

For these games some of the previous results generalize immediately. For example, the same argument as before shows that Breaker cannot have a winning strategy. Suppose that the hypergraph H has finite edges only, then the weak infinite game played on H is determined. The claim of Proposition 6 remains valid, too: This game is a win for Maker if and only if, for some finite subhypergraph $G \subset H$, the (finite) weak game on G is a win for Maker. In other words, if Maker has a winning strategy in a weak infinite game, then for some natural number n, he can win within n moves. Observe that the proof of the original claim does not work here since it says nothing about the cases when Maker wins after infinitely many moves.

The case is far from this in strong games. Let $\kappa(\alpha)$ denote the supremum of cardinals below the orinal number α; this $\kappa(\alpha)$ is always a cardinal, and, e.g., $\kappa(\omega) = \kappa(\omega + \omega) = \kappa(\omega_1) = \aleph_0$.

Theorem 15. *Let α be a limit ordinal. There exists a hypergraph H with finite edges and set of vertices of cardinality $\kappa(\alpha)$ such that Maker wins the strong infinite game on H before his* αth *move, but for every $\beta < \alpha$, Breaker* *has such a counterplay which postpones his defeat until after the* β*th move.* ∎

The cardinality of the board in these examples is the smallest possible since for every cardinal $\kappa < \alpha$, Breaker must be able to pick κ different vertices.

While weak infinite games are always determined, the determinacy of strong games may depend on various set-theoretical assumptions. First of all, there exists a hypergraph H whose edges are finite sets of reals such that the strong infinite game played on H is undetermined. What is more, if either Maker or Breaker plays by a strategy, then the other can win against that strategy before the $(\omega + 20)$th move. Therefore, this game gives an example of an undetermined strong game of length $\omega + \omega$. The board of this example is the set of reals, and one may ask for an undetermined game with countable board.

THEOREM 16. (i) *Assume there is a measurable cardinal. Then every strong amoeba game on a countable board with finite edges and of length* $\omega + \omega$ *is determined.* (ii) *Assume the axiom of constructibility* $V = L$. *Then there is an undetermined game of this type.*

Proof. Part (i) is a consequence of Martin's result on analytic games [18]. The construction in (ii) uses the following idea. In the first ω moves the players build two trees, one for Maker and one for Breaker. On their next ω moves they are forced to climb up on some branch of their own tree. Now if either of them playes by a strategy, then the other can arrange that his tree has an infinite branch and the opponent's has none. (Here is the point where the assumption $V \doteq L$ is used.) The one who has no infinite branch, runs out of his tree eventually, and loses the game. Of course, a lot of auxiliary edges of the hypergraph of this game force the players to follow this pattern. ∎

The Case of Countable Edges

From now on we shall deal with weak games only, and the edges of the hypergraphs are required to be infinite. The simplest case is when all the edges are countable, and the board is also countable. If the hypergraph has also countably many edges, then the game is a win for Breaker: in his nth move he can pick a new vertex from the nth edge. One can easily construct a game of this type with continuum many edges which is a win for Maker. There are undetermined games here, too. The following example was found independently by R. McKenzie and J. Paris; the proof given here is due to McKenzie.

THEOREM 17. *Suppose that the edges of a hypergraph form a nontrivial ultrafilter on* ω. *The weak infinite game played on it is undetermined.*

Proof. Suppose first that Breaker has a winning strategy. When a play ends, all the elements of the board are occupied, so either the set of Maker's points, or that of Breaker's points is in the given ultrafilter U, but not both. Now if Breaker plays by his winning strategy, then at the end, his points form an element of U. But Maker can also play by this strategy which ensures that he also covers some elements of U, a contradiction.

Now suppose Maker has a winning strategy, and let them play three instances of the game as shown in Fig. 3. Let i denote the number of square Maker is advised by his strategy to occupy first. In his first move Maker occupies all squares in the row numbered by i, then Breaker occupies one square and Maker answers in the same row by his strategy, etc. Breaker may arrange that after the first ω moves every square is occupied, and in every column of three squares, except for the ith one, at least one square belongs to him. But Maker won the game in each row, therefore his squares in the rows are elements of U. So their intersection must be infinite, a contradiction. ∎

An undetermined game on a countable board cannot have fewer edges than in this example, namely continuum. This is because Martin's axiom implies that in such games Breaker wins within the first ω moves. On the other hand the existence of a game with countable board and fewer than continuum many edges in which Breaker has no winning strategy is consistent with ZFC.

The ultrafilter game can be made biased by allowing Maker and Breaker to pick m and b vertices, respectively, in a move. Theorem 17 says that if $m = b = 1$, then for no nontrivial ultrafilter does Maker have a winning strategy. Fred Galvin proved that if $m \geqq 2b$, then for some nontrivial ultrafilter Maker can win by a strategy, but if $b \geqq m$, then there is no such ultrafilter. For the missing cases $b < m < 2b$ nothing is known.

Several other generalizations were discussed by S. Hechler and R. McKenzie. For example, a player is allowed to pick any finite number of vertices, or even infinitely many if they are not in the ultrafilter. They characterize in many cases the ultrafilters for which Maker has a winning strategy.

If we raise the cardinality of the board, new problems arise. The simple of them is yet unsolved. Is there a weak infinite amoeba game with $\aleph_1$ countable edges which is a win for Maker? The existence of such a game can be shown to be consistent.

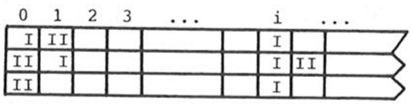

FIGURE 3

We call a weak infinite game *fat* if the edges are countable, Maker has a winning strategy, but Breaker wins whenever Maker restricts himself to picking points from a fixed small subset of the vertices. Here *small* means that the cardinality of the subset of strictly less than the cardinality of the board. One can construct a fat game with $\aleph_1$ vertices easily, and A. Hajnal has proved that the axiom of constructibility implies the existence of fat games with k vertices for every not weakly compact cardinal $\kappa \geqslant \aleph_1$. It is not known whether the nonexistence of a fat game with $\aleph_2$ vertices is consistent.

Infinite Ramsey Game

Let κ and λ be infinite cardinals, and let $k \geqq 2$ be a natural number. Recall that $[S]^k$ denotes the family of all k-element subsets of S, so $[\kappa]^2$ can be regarded as a set of edges of a complete graph with κ vertices. The weak infinite amoeba game $R_k(k, \lambda)$ is an immediate generalization of the finite Ramsey games discussed in the first section. The players alternately occupy k-element subsets of κ (i.e., elements of $[\kappa]^k$) as long as there remain any, and Maker wants to pick all k-element subsets of some $X \subset \kappa$ of cardinality λ.

In the game $R_{<\omega}(\kappa, \lambda)$ the players may pick any finite subset of κ (including the empty set), and Maker's aim is to pick all finite subsets of some $X \subset \kappa$ of cardinality λ. In this game the only reasonable starting move of Maker is to occupy the empty set, otherwise he cannot win the game.

The chromatic number of the hypergraph of a finite Ramsey game gives some insight into the structure of the game. The chromatic numbers of these infinite hypergraphs were investigated thoroughly, and they are closely related to the so-called partition relations introduced by Erdös. The partition relation $\kappa \to (\lambda)_r^k$, where κ and λ are infinite cardinals, k and r are natural numbers, means the following assertion: Whenever the k-element subsets of κ (i.e., elements of $[\kappa]^k$) are colored with r colors, then for some subset $X \subset \kappa$ of cardinality λ, all the k-element subsets of X have the same color. The meaning of the partition relation $\kappa \to (\lambda)_r^{<\omega}$ is similar. Whenever the finite subsets of κ are colored with r colors, then there exists a subset $X \subset \kappa$ of cardinality λ such that for each natural number k, all k-element subsets of X have the same color (but this color may depend on k).

In a well-known theorem of Erdös and Radó [12] it is stated that the relations $(2^\kappa)^+ \to (\kappa^+)_2^2$, $(2^{2^\kappa})^+ \to (\kappa^+)_2^3$, etc., hold for every infinite cardinal κ. Observe that these claims are equivalent with the statements that the chromatic numbers of the hypergraphs of the games $R_2((2^\kappa)^+, \kappa^+)$, $R_3((2^{2^\kappa})^+, \kappa^+)$, etc., are at least 3. Unfortunately, while in the finitistic case the game $R_k(N, n)$ was known to be determined, and consequently, the large chromatic number implies the existence of a winning strategy for Maker, for infinite Ramsey games the determinacy is by no means a triviality. Nagy observed, however, that the proofs of these partition relations can be turned to yield winning strategies for Maker, so the games listed above are wins for Maker.

For the case $\kappa = \lambda = \aleph_0$, Baumgartner *et al.* in a joint paper [2] proved that Maker has winning strategies in the games $R_k(\aleph_0, \aleph_0)$. For uncountable κ and λ the case $k = 2$ is solved almost completely under the assumption GCH. The results are due to Hajnal and Nagy.

THEOREM 18. *Assume* GCH. *For every infinite cardinal number* κ, *Maker has a winning strategy in the game* $R_2(\kappa^{++}, \kappa^{+})$ *and Breaker wins the game* $R_2(\kappa^{+}, \kappa^{+})$. *If* κ *is a singular cardinal (i.e.,* κ *can be partitioned into fewer than* κ *parts so that each part has cardinality less than* κ*), then the game* $R_2(\kappa, \kappa)$ *is a win for Maker.* ∎

If GCH holds, then, as we have remarked above, Maker wins the game $R_3(\aleph_3, \aleph_1)$. R. Laver proved that the existence of a winning strategy for Breaker in $R_3(\aleph_2, \aleph_1)$ is consistent. At present no more is known about these games.

In [2], Baumgartner *et al.* showed that Breaker has a winning strategy in the game $R_{<\omega}(\aleph_0, \aleph_0)$. Hajnal and Nagy proved that in the presence of the axiom of constructibility, the game $R_{<\omega}(\kappa, \lambda)$ is determined, and Maker has a winning strategy if and only if the partition relation $\kappa \rightarrow (\lambda)_2^{<\omega}$ holds.

A Game of P. Erdös

Finally, we cannot resist mentioning a result of Galvin and Nagy (independently) concerning a biased game of Erdös. In this game Maker and Breaker pick unpicked real numbers alternately, Maker picks one, and Breaker picks countable many in a single move. The game ends if every real has been chosen, and Maker wants to pick elements of a long arithmetic progression. Evidently he can pick an A.P. of one or two terms, and Breaker has a strategy which prevents Maker from picking all terms of an infinite A. P. For a natural number n, denote by E_n the game in which Maker wins by picking an A.P. of $n + 1$ terms. In these games either Maker or Breaker has a winning strategy depending on where the cardinality of the reals occurs in the sequence of cardinals.

THEOREM 19. *In the game* E_n, *if* $2^{\aleph_0} \geqq \aleph_n$, *then Maker; if* $2^{\aleph_0} < \aleph_n$, *then Breaker has a winning strategy.* ∎

The game as well as the theorem generalizes easily for arbitrary vector spaces over rationals; in the theorem, $2^{\aleph_0}$ should be replaced by the cardinality of the vector space.

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References

1. M. Ajtai, L. Csirmaz, and Zs. Nagy, On a generalization of the game Go-Moku, I–II, *Acta Math. Acad. Sci. Hungar.* to appear.
2. J. Baumgartner, F. Galvin, R. Laver, and R. McKenzie, Game theoretic versions of partition relations, *in* "Infinite and Finite Sets," Vol. I, pp. 131–135, Colloq. Keszthely, 1973.
3. J. Beck, On positional games, *J. Combin. Theory, Ser. A* **30** (1981), 117–133.
4. J. Beck, Van der Waerden and Ramsey type games, *Combinatorica* **1** (1981), 103–116.
5. J. Beck, Remarks on biased positional games, *Acta Math. Acad. Sci. Hungar*, to appear.
6. C. Berge, Sur les jeux positionelles, Cahiers Centre Études *Rech. Opér.* **18** (1976).
7. N. G. de Bruijn and P. Erdös, A colour problem for infinite graphs, *Indag. Math.* **13** (1951), 371–373.
8. V. Chvátal and P. Erdös, Biased positional games, *Ann. Discrete Math.* **2** (1978), 221–229.
9. H. E. Dudeney, "The Canterbury Puzzles," Dover, New York, 1958.
10. P. Erdös, A. Hajnal, and R. Radó, Partition relations for cardinal numbers, *Acta Math. Acad. Sci. Hungar.* **16** (1965), 93–196.
11. P. Erdös and L. Lovász, Problems and results on 3-chromatic hypergraphs and some related questions, *in* "Infinite and Finite Sets," Vol. II, pp. 609–627, Colloq. Keszthely, 1973.
12. P. Erdös and R. Radó, Combinatorial theorems on classifications of subsets of a given set, *Proc. London Math. Soc.* **3** (1952), 417–493.
13. P. Erdös and J. Selfridge, On a combinatorial game, *J. Combin. Theory, Ser. A* **14** (1973), 298–301.
14. D. Gale and F. M. Stewart, Infinite games of perfect information, Contributors to the theory of games, *Ann. Math. Stud.* (1953), 245–266.
15. A. W. Hales and R. I. Jewett, On regularity and positional games, *Trans. Amer. Math. Soc.* **106** (1963), 222–229.
16. H. J. Keisler, Infinite quantifiers and continuous games, *in* "Applications of Model Theory to Algebra, Analysis and Probability," pp. 228–264, Holt, Rinehart & Winston, New York, 1969.
17. A. Lehman, A solution to the Shannon switching game, *SIAM J. Appl. Math.* **12** (1964), 687–725.
18. D. A. Martin, Measurable cardinals and analytic games, *Fund. Math.* **66** (1970), 287–291.
19. C. St. J. A. Nash-Williams, Edge-disjoint spanning trees of finite graphs, *J. London Math. Soc.* **36** (1961), 445–450.
20. J. L. Paul, Tic-Tac-Toe in n-dimensions, *Math. Mag.* **51** (1978), 45–49.
21. W. T. Tutte, On the problem of decomposing a graph into n connected factors, *J. London Math. Soc.* **36** (1961), 221–230.

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